

SOME EINSTEIN HOMOGENEOUS RIEMANNIAN FIBRATIONS

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ABSTRACT. We study the existence of projectable G -invariant Einstein metrics on the total space of G -equivariant fibrations $M = G/L \rightarrow G/K$, for a compact connected semisimple Lie group G . We obtain necessary conditions for the existence of such Einstein metrics in terms of appropriate Casimir operators, which is a generalization of the result by Wang and Ziller about Einstein normal metrics. We describe binormal Einstein metrics which are the orthogonal sum of the normal metrics on the fiber and on the base. The special case when the restriction to the fiber and the projection to the base are also Einstein is also considered. As an application, we prove the existence of a non-standard Einstein invariant metric on the Kowalski n -symmetric spaces.

1. INTRODUCTION

We describe a class of invariant Einstein metrics on a homogeneous manifold $M = G/L$ of a compact connected semisimple Lie group G , which is consistent with a homogeneous fibration $G/L \rightarrow G/K$.

A Riemannian metric g is said to be Einstein if its Ricci curvature satisfies the Einstein equation $Ric = Eg$, for some constant E . The Einstein equation is a system of partial differential equations of second order, which is in general unmanageable. A few results about Einstein metrics are known in the general case and many results are known under some extra assumptions (special type of metrics or metric with large isometry group). For example, there are deep results about Kähler-Einstein ([29], [2], [23], [24]) and Sasakian-Einstein manifolds ([7]). For a homogeneous space the Einstein equation reduces to a system of algebraic equations, which is still very complicated. Even for homogeneous spaces we are far from knowing a full description. For instance, homogeneous Einstein metrics on spheres and projective spaces were classified by Ziller ([31]) and Einstein normal homogeneous manifolds were classified by Wang and Ziller ([25]). Every isotropy irreducible space, in particular any irreducible symmetric space ([10],[14]) is an Einstein manifold ([5], [28]). Recently homogeneous Einstein metrics on homogeneous spaces with exactly two isotropy summands were classified by Dickinson and Kerr ([9]). Nowadays, it is known that every compact simply-connected homogeneous manifold with dimension less or equal to 11 admits a homogeneous Einstein metric: a 2 or 3-dimensional manifold has constant sectional curvature ([5]); in dimension 4, the result was shown by Jensen ([12]), and by Alekseevsky, Dotti and Ferraris in dimension 5 ([1]); in dimension 6, the result is due to Nikonov and Rodionov

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([18]), and in dimension 7 it is due to Castellani, Romans and Warner ([8]). All the 7-dimensional homogeneous Einstein manifolds ([17]) were obtained by Nikonorov. These results were extended to dimension up to 11 by Böhm and Kerr ([6]). All these results deal with the case of positive Einstein constant. Any homogeneous Riemannian manifold with zero constant is locally flat. There are also many results about invariant Einstein metrics with negative Einstein constant on solvmanifolds.

Einstein homogeneous fibrations have also been the object of study. We recall, for instance, the work of Jensen on principal fibers bundles ([13]) and the work of Wang and Ziller on principal torus bundles ([27]).

This paper is devoted to investigation of G -invariant Einstein metrics on the total space of G -equivariant fibrations. Let G be a compact connected semisimple Lie group and $L \subsetneq K \subsetneq G$ connected closed non-trivial subgroups of G . We consider the fibration

$$(1.1) \quad M = G/L \rightarrow G/K = N, \text{ with fiber } F = K/L$$

and investigate the existence of G -invariant Einstein metrics on M such that the natural projection $M \ni aL \mapsto aK \in N$, $a \in G$, is a Riemannian submersion with totally geodesic fibers.

By \mathfrak{g} , \mathfrak{k} and \mathfrak{l} we denote the Lie algebras of G , K and L , respectively. By Φ and $\Phi_{\mathfrak{k}}$ we denote the Killing forms of G and K , respectively. We set $B = -\Phi$ and since G is compact and semisimple, B is positive definite. We denote by $B_{\mathfrak{q}}$ the restriction of B to some subspace $\mathfrak{q} \subset \mathfrak{g}$.

We consider a B -orthogonal decomposition of \mathfrak{g} given by

$$(1.2) \quad \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n},$$

where $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ and $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$ are reductive decompositions for M , N and F , respectively. An $Ad K$ -invariant Euclidean product on \mathfrak{n} induces a G -invariant metric g_N on N and an $Ad L$ -invariant Euclidean product on \mathfrak{p} induces a G -invariant metric g_F on F . The orthogonal direct sum of the metric g_F on the fiber and the metric g_N on the base space N defines a G -invariant metric g_M on M which projects onto a G -invariant metric g_N on N . We recall the following result due to L. Bérard-Bergery:

Theorem 1.1. ([4], [5, 9 §H]) *Let $M = G/L \rightarrow G/K = N$ be a G -equivariant fibration, for a Lie group G , where K , L are compact subgroups. Let g_M be the G -invariant metric on M given by the orthogonal sum of a G -invariant metric g_N on N and a G -invariant metric g_F on $F = K/L$. The natural projection $M \ni aL \mapsto aK \in N$ is a Riemannian submersion from (M, g_M) to (N, g_N) with totally geodesic fibers.*

Moreover, if \mathfrak{p} and \mathfrak{n} do not contain any equivalent $Ad L$ -submodules, then any G -invariant metric with totally geodesic fibers is constructed in this fashion.

Throughout this paper, a G -invariant metric on M such that the natural projection $M \ni aL \mapsto aK \in N$ is a Riemannian submersion with totally geodesic fibers shall be called an **adapted** metric.

An adapted metric on M shall be denoted by g_M ; g_N shall denote the projection of g_M onto the base space N and g_F its restriction to the fiber F . By using the

B -orthogonal decomposition of \mathfrak{g} fixed in (1.2), we denote by $g_{\mathfrak{m}}$, $g_{\mathfrak{n}}$ and $g_{\mathfrak{p}}$ the invariant Euclidean products on \mathfrak{m} , \mathfrak{n} and \mathfrak{p} , respectively, which determine g_M , g_N and g_F . Whereas $g_{\mathfrak{m}}$ and $g_{\mathfrak{p}}$ are $Ad L$ -invariant, $g_{\mathfrak{n}}$ is $Ad K$ -invariant.

We consider a B -orthogonal decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_s$ of \mathfrak{p} into irreducible $Ad L$ -submodules and a B -orthogonal decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_n$ of \mathfrak{n} into irreducible $Ad K$ -submodules. Throughout we assume the following hypothesis:

- (1.3) $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are pairwise inequivalent irreducible $Ad L$ -submodules;
 $\mathfrak{n}_1, \dots, \mathfrak{n}_n$ are pairwise inequivalent irreducible $Ad K$ -submodules;
 \mathfrak{p} and \mathfrak{n} do not contain equivalent $Ad L$ -submodules.

We remark that the submodules \mathfrak{n}_j are not required to be $Ad L$ -irreducible. Under the hypothesis (1.3), according to Schur's Lemma, any $Ad L$ -invariant scalar product on $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n}$ which restricts to an $Ad K$ -invariant scalar product on \mathfrak{n} is of the form

$$(1.4) \quad g_{\mathfrak{m}} = \underbrace{(\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a})}_{g_{\mathfrak{p}}} \oplus \underbrace{(\oplus_{j=1}^n \mu_j B_{\mathfrak{n}_j})}_{g_{\mathfrak{n}}}, \quad \lambda_a, \mu_j > 0$$

Since an adapted metric g_M on M projects onto a G -invariant metric on N , g_M is necessarily induced by an Euclidean product $g_{\mathfrak{m}}$ of the form (1.4).

Throughout this paper any homogeneous fibration $M \rightarrow N$ and any adapted metric g_M on M are as defined above. In Section 3 we derive formulae for the Ricci curvature for an adapted metric g_M and find some necessary conditions that g_M is Einstein. A main result consists of necessary conditions for existence of an Einstein adapted metric described just in terms of algebraic conditions on the Casimir operators of the isotropy submodules \mathfrak{p}_a and \mathfrak{n}_k . We recall that if U is a vector subspace of \mathfrak{g} , the Casimir operator of U (with respect to the Killing form Φ) is the operator

$$(1.5) \quad C_U = \sum_i ad_{u_i} ad_{u'_i} \in \mathfrak{gl}(\mathfrak{g}),$$

where $\{u_i\}_i$ and $\{u'_i\}_i$ are bases of U which are dual with respect to Φ , i.e., $\Phi(u_i, u'_j) = \delta_{ij}$. More precisely, we prove the following result:

Theorem 1.2. *Let $M = G/L \rightarrow G/K$ be a homogeneous fibration. If there exists on M an Einstein adapted metric, then*

- (i) *there are positive constants $\lambda_1, \dots, \lambda_s$ such that $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}$ is scalar on each \mathfrak{n}_j , where $C_{\mathfrak{p}_a}$ is the Casimir operator of \mathfrak{p}_a ;*
- (ii) *if g_N is not a multiple of B , then there are positive constants ν_1, \dots, ν_n , which are not all equal, such that $\sum_{j=1}^n \nu_j C_{\mathfrak{n}_j}(\mathfrak{p}) \subset \mathfrak{k}$, where $C_{\mathfrak{n}_j}$ is the Casimir operator of \mathfrak{n}_j .*

In the following sections we focus in some special cases. In Section 4, we consider the special case when the adapted metric is binormal. A **binormal** metric on M is a G -invariant metric g_M such that its restrictions to the fiber and projection onto the base, g_F and g_N , are multiples of the restrictions of the Killing form of G , i.e., g_M is defined by the Euclidean product

$$(1.6) \quad g_{\mathfrak{m}} = \lambda B_{\mathfrak{p}} \oplus \mu B_{\mathfrak{n}}.$$

A binormal metric is clearly an adapted metric. Binormal metrics are a very natural class of metrics on homogeneous fibrations which generalize normal metrics. We recall that a normal metric on M is defined by the restriction of an AdG -invariant positive definite symmetric bilinear map to \mathfrak{m} . Einstein normal homogeneous manifolds were classified by Wang and Ziller ([25]) and Rodionov ([21]). A necessary condition for existence of an Einstein normal metric on M is that the Casimir operator of \mathfrak{l} is scalar on \mathfrak{n} ([25]). Similarly, Theorem 1.2 states that a necessary condition for existence of an Einstein binormal metric is that the Casimir operator of the tangent space to the fibers, $C_{\mathfrak{p}}$, is scalar on each of the irreducible horizontal submodules \mathfrak{n}_j . If this condition is satisfied, then the following theorem reduces the problem of existence of Einstein binormal metrics to a system of quadratic equations with one variable. Thus, if $C_{\mathfrak{p}}$ is scalar on each \mathfrak{n}_j , the only difficulty deciding about the existence of an Einstein binormal metric is the computation of the necessary coefficients.

Theorem 1.3. *Let $M = G/L \rightarrow G/K$ be a homogeneous fibration.*

(i) *If the Casimir operator of \mathfrak{p} , $C_{\mathfrak{p}}$, is not scalar on some \mathfrak{n}_j , then there are no Einstein binormal metrics on M ;*

(ii) *Suppose that $C_{\mathfrak{p}}$ is scalar on each \mathfrak{n}_j , i.e., $C_{\mathfrak{p}}|_{\mathfrak{n}_j} = b^j Id$. Then there is a one-to-one correspondence, up to homothety, between Einstein binormal metrics on M and positive solutions of the following system of quadratic equations on the unknown $X \in \mathbb{R}$:*

$$(1.7) \quad \delta_{ij}^{\mathfrak{t}}(1 - X) = \delta_{ij}^{\mathfrak{l}}, \text{ if } n > 1,$$

$$(1.8) \quad (2\delta_{ab}^{\mathfrak{l}} + \delta_{ab}^{\mathfrak{t}})X^2 = \delta_{ab}^{\mathfrak{t}}, \text{ if } s > 1,$$

$$(1.9) \quad (\gamma_a + 2c_{\mathfrak{l},a})X^2 - (1 + 2c_{\mathfrak{t},j})X + (1 - \gamma_a + 2b^j) = 0,$$

for $a, b = 1, \dots, s$ and $i, j = 1, \dots, n$. Here $c_{\mathfrak{l},a}$ is the eigenvalue of $C_{\mathfrak{l}}$ on \mathfrak{p}_a , γ_a is the constant determined by

$$\Phi_{\mathfrak{t}}|_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a},$$

$c_{\mathfrak{t},j}$ is the eigenvalue of $C_{\mathfrak{t}}$ on \mathfrak{n}_j and the δ 's are the differences $\delta_{ij}^{\mathfrak{t}} = c_{\mathfrak{t},i} - c_{\mathfrak{t},j}$, $\delta_{ij}^{\mathfrak{l}} = c_{\mathfrak{l},i} - c_{\mathfrak{l},j}$, $\delta_{ab}^{\mathfrak{t}} = \gamma_a - \gamma_b$ and $\delta_{ab}^{\mathfrak{l}} = c_{\mathfrak{l},a} - c_{\mathfrak{l},b}$.

If such a positive solution X exists, then Einstein binormal metrics are, up to homothety, given by

$$g_{\mathfrak{m}} = B_{\mathfrak{p}} \oplus XB_{\mathfrak{n}}.$$

In particular, this allows us to characterize Einstein adapted metrics on fibrations such that the fiber and base spaces are isotropy irreducible spaces. We remark that the fact that the base space N is isotropy irreducible does not imply that M has only two irreducible isotropy subspaces, since the horizontal subspace \mathfrak{n} is AdK -irreducible but is not in general AdL -irreducible. The particular case of existence of G -invariant Einstein metrics when M has exactly two irreducible isotropy subspaces was studied by Wang and Ziller, under some assumptions, in [26] and more recently, in full generality, by Dickinson and Kerr in [9], who classified all such metrics.

Corollary 1.4. *Let $M = G/L \rightarrow G/K = N$ be a homogeneous fibration. Suppose that the fiber $F = K/L$ and the base space N are isotropy irreducible spaces and $\dim F > 1$. There exists on M an Einstein adapted metric if and only if the following two conditions are satisfied:*

- (i) $C_{\mathfrak{p}}$ is scalar on \mathfrak{n} ;
- (ii) $\Delta \geq 0$, where

$$\Delta = (1 + 2c_{\mathfrak{t},\mathfrak{n}})^2 - 4(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})(1 - \gamma + 2b),$$

$c_{\mathfrak{t},\mathfrak{n}}$ is the eigenvalue of $C_{\mathfrak{t}}$ on \mathfrak{n} , $c_{\mathfrak{l},\mathfrak{p}}$ is the eigenvalue of $C_{\mathfrak{l}}$ on \mathfrak{p} , b is the eigenvalue of $C_{\mathfrak{p}}$ on \mathfrak{n} and γ is such that $\Phi_{\mathfrak{t}}|_{\mathfrak{p} \times \mathfrak{p}} = \gamma \Phi|_{\mathfrak{p} \times \mathfrak{p}}$.

If these conditions are satisfied, then Einstein adapted metrics are, up to homothety, given by

$$g_{\mathfrak{m}} = B_{\mathfrak{p}} \oplus XB_{\mathfrak{n}}, \text{ where } X = \frac{1 + 2c_{\mathfrak{t},\mathfrak{n}} \pm \sqrt{\Delta}}{2(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})}.$$

In the case when F is 1-dimensional, the fibration $M \rightarrow N$ is a principal circle bundle, since F is an abelian compact connected 1-dimensional group. We recall that Einstein metrics on principal fiber bundles have been widely studied ([13],[27]) and, in particular, homogeneous Einstein metrics on circle bundles were classified McKenzie Y. Wang and Wolfgang Ziller in [27]. From Theorem 1.3, We obtain the following well known result obtained previously in [27].

Corollary 1.5. *Let $M = G/L \rightarrow G/K = N$ be a homogeneous fibration. Suppose that N is isotropy irreducible and the fiber $F = K/L$ is isomorphic to the circle group. There exists on M exactly one G -invariant Einstein metric, up to homothety, given by*

$$g_{\mathfrak{m}} = B_{\mathfrak{p}} \oplus XB_{\mathfrak{n}}, \text{ where } X = \frac{2 + m}{m(1 + 2c_{\mathfrak{t},\mathfrak{n}})},$$

$c_{\mathfrak{t},\mathfrak{n}}$ is the eigenvalue of $C_{\mathfrak{t}}$ on \mathfrak{n} and $m = \dim N$.

Also interesting necessary conditions are found if the fiber is not isotropy irreducible.

Corollary 1.6. *Let $M = G/L \rightarrow G/K$ be a homogeneous fibration. Suppose the fiber $F = K/L$ is not isotropy irreducible and there exists a constant α such that*

$$(1.10) \quad \Phi \circ C_{\mathfrak{l}}|_{\mathfrak{p} \times \mathfrak{p}} = \alpha \Phi_{\mathfrak{t}}|_{\mathfrak{p} \times \mathfrak{p}}.$$

For $a = 1, \dots, s$, let γ_a be the constant defined by $\Phi_{\mathfrak{t}}|_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a}$. If $\gamma_a \neq \gamma_b$, for some a, b , then there exists an Einstein binormal metric on M if and only if $C_{\mathfrak{p}}$ is scalar on each \mathfrak{n}_j and

$$(1.11) \quad c_{\mathfrak{l},j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right) \left(c_{\mathfrak{t},j} + \frac{1}{2}\right), \quad j = 1, \dots, n,$$

where $c_{\mathfrak{l},j}$ and $c_{\mathfrak{t},j}$ are the eigenvalues of $C_{\mathfrak{l}}$ and $C_{\mathfrak{t}}$, respectively, on \mathfrak{n}_j . In this case, there is a unique binormal Einstein metric, up to homothety, given by

$$g_{\mathfrak{m}} = B_{\mathfrak{p}} \oplus \frac{1}{\sqrt{2\alpha + 1}} B_{\mathfrak{n}}$$

and, furthermore, the number $\sqrt{2\alpha + 1}$ is rational.

Interesting applications arise from the result above, for instance, when the fiber $F = K/L$ is a symmetric space, since, in this case, $\alpha = 1/2$ and clearly $\sqrt{2\alpha + 1}$ is not rational. This implies that if an Einstein binormal metric exists, then the Casimir operator of \mathfrak{k} must be scalar on \mathfrak{p} . Though, Einstein adapted metrics on homogeneous fibrations with symmetric fiber are out of scope of this paper, the reader is invited to find such applications in [3].

A natural question is to determine an Einstein adapted metric whose restriction to the fiber and projection to the base space are also Einstein metrics. This problem has been approached by several authors, see for instance, results by Berard-Bergery, Matsuzawa and Wang and Ziller in [5] and [25]. In Section 5 we prove the following two results:

Theorem 1.7. *Let g_M be an Einstein adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K = N$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

If g_N and g_F are also Einstein, then

$$(1.12) \quad \frac{\mu_j}{\mu_k} = \left(\frac{b^j}{b^k} \right)^{\frac{1}{2}}, \quad j, k = 1, \dots, n,$$

$$(1.13) \quad \frac{\lambda_a}{\lambda_b} = \sum_{j=1}^n \frac{C_{\mathfrak{n}_j, b}}{b_j} \bigg/ \sum_{j=1}^n \frac{C_{\mathfrak{n}_j, a}}{b_j}, \quad a, b = 1, \dots, s,$$

where b^j is the eigenvalue of $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}$ on \mathfrak{n}_j and $c_{\mathfrak{n}_j, a}$ is defined by

$$\Phi(C_{\mathfrak{n}_j} \cdot, \cdot) |_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{\mathfrak{n}_j, a} \Phi |_{\mathfrak{p}_a \times \mathfrak{p}_a}.$$

In particular, there exists at most one G -invariant metric g_N on N and one K -invariant metric g_F on F such that g_M is Einstein.

Theorem 1.8. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K = N$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

Suppose that g_M , g_N and g_F are Einstein and let E , E_F and E_N be the corresponding Einstein constants. If $E \neq E_N$, then

$$\mu_j = \left(\frac{b^j}{2(E_N - E)} \right)^{\frac{1}{2}},$$

$$\lambda_a = 2 \frac{E - E_F}{E_N - E} \left(\sum_{j=1}^n \frac{C_{\mathfrak{n}_j, a}}{b^j} \right)^{-1}.$$

where b^j is the eigenvalue of the operator $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}$ on \mathfrak{n}_j and $c_{\mathfrak{n}_j, a}$ is defined by $\Phi(C_{\mathfrak{n}_j} \cdot, \cdot) |_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{\mathfrak{n}_j, a} \Phi |_{\mathfrak{p}_a \times \mathfrak{p}_a}$.

Finally, in Section 6, we prove the existence of a non-standard Einstein metric on the n -symmetric spaces $M = \frac{G_0^n}{\Delta^n G_0}$, for a compact connected simple Lie group G_0 , where $\Delta^n G_0$ is the diagonal subgroup. We recall that Wang and Ziller have shown

that the standard metric on these spaces is Einstein ([26], [21]). To investigate the existence of a non-standard Einstein G -invariant metric on M , we consider the fibration

$$\frac{G_0^n}{\Delta^n G_0} \rightarrow \frac{G_0^p}{\Delta^p G_0} \times \frac{G_0^q}{\Delta^q G_0}, \text{ with fiber } F = \frac{\Delta^p G_0 \times \Delta^q G_0}{\Delta^n G_0},$$

for $n = p + q$. The vertical isotropy subspace \mathfrak{p} is $Ad L$ -irreducible. However, the horizontal isotropy subspace \mathfrak{n} is not $Ad K$ -irreducible. We consider a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$,

$$(1.14) \quad \begin{aligned} \mathfrak{n}_1 &= \{(X_1, \dots, X_p, 0, \dots, 0) : X_j \in \mathfrak{g}_0, \sum X_j = 0\} \subset \mathfrak{g}_0^p \times 0_q \\ \mathfrak{n}_2 &= \{(0, \dots, 0, X_1, \dots, X_q) : X_j \in \mathfrak{g}_0, \sum X_j = 0\} \subset 0_p \times \mathfrak{g}_0^q, \end{aligned}$$

where \mathfrak{g}_0 is the Lie algebra of G_0 . We remark that this decomposition of \mathfrak{n} is not unique and $\mathfrak{n}_1, \mathfrak{n}_2$ are not $Ad K$ -irreducible either. Hence, the hypothesis (1.3) is not satisfied. We can still consider an adapted metric g_M defined by a scalar product of the form

$$(1.15) \quad g_{\mathfrak{m}} = \lambda B_{\mathfrak{p}} \oplus \mu_1 B_{\mathfrak{n}_1} \oplus \mu_2 B_{\mathfrak{n}_2},$$

but we should keep in mind that other adapted metrics might exist which are not of this form. We prove the following result:

Theorem 1.9. *Let G_0 be a compact connected simple Lie group and consider the fibration*

$$M = \frac{G_0^n}{\Delta^n G_0} \rightarrow \frac{G_0^p}{\Delta^p G_0} \times \frac{G_0^q}{\Delta^q G_0},$$

where $p + q = n$ and $2 \leq p \leq q \leq n - 2$. For $n = 4$, the only Einstein adapted metric is the standard metric. If $n > 4$, there exist on M at least one non-standard Einstein adapted metric of the form (1.15). This non-standard Einstein adapted metric is binormal if and only if $p = q$. Furthermore, its projection onto the base space is also Einstein if and only if $p = q$.

2. THE RICCI CURVATURE

Let $M = G/L$ be a homogeneous manifold of a connected Lie group G , where L is a compact subgroup. Let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ be a reductive decomposition of \mathfrak{g} . In this section we describe the Ricci curvature, Ric , of the G -invariant metric g_M on M associated to the $Ad L$ -invariant scalar product \langle, \rangle on \mathfrak{m} . For $X \in \mathfrak{g}$, let P_X and T_X be the endomorphisms of \mathfrak{m} defined by

$$(2.1) \quad P_X Y = [X, Y]_{\mathfrak{m}}, Y \in \mathfrak{m}$$

$$(2.2) \quad \langle T_X Y, Z \rangle = \langle X, P_Y Z \rangle, Y, Z \in \mathfrak{m}$$

where the subscript \mathfrak{m} denotes projection onto \mathfrak{m} . For $X \in \mathfrak{g}$, The *Nomizu operator* L_X of the scalar product \langle, \rangle ([19], [15]) is defined by

$$(2.3) \quad L_X Y = -\nabla_Y X^*, Y \in \mathfrak{m}$$

where ∇ is the Riemannian connection of g_M and X^* is the Killing vector field generated by X . We have

$$(2.4) \quad L_X Y = \frac{1}{2} P_X Y + U(X, Y),$$

where $U : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ is the operator

$$(2.5) \quad U(X, Y) = -\frac{1}{2}(T_X Y + T_Y X), \quad X, Y \in \mathfrak{m}.$$

Moreover, L_X is skew-symmetric with respect to \langle, \rangle and $L_X Y - L_Y X = P_X Y$, $Y \in \mathfrak{m}$. The metric g_M is called *naturally reductive* if $U = 0$. The curvature tensor of g_M at the point $o = eL$ can be written as

$$(2.6) \quad R(X, Y) = [L_X, L_Y] - L_{[X, Y]}, \quad X, Y \in \mathfrak{m} = T_o M.$$

The sectional curvature K of g_M is defined by

$$(2.7) \quad K(Z, X) = \langle R(Z, X)X, Z \rangle,$$

for every $X, Z \in \mathfrak{m}$ orthonormal with respect to \langle, \rangle . The Ricci curvature of g_M is determined by

$$(2.8) \quad Ric(X, X) = \sum_i K(Z_i, X), \quad X \in \mathfrak{m}$$

where $(Z_i)_i$ is an orthonormal basis of \mathfrak{m} with respect to \langle, \rangle . The metric g_M is said to be an Einstein metric if there exists a constant E such that $Ric = E g_M$. Below we describe the Ricci curvature of the G -invariant metric g_M by using the endomorphisms described above. The proof is out of the scope of this paper and can be found in detail in [3, §1.1].

Lemma 2.1. [3, §1.1] *Let $X, Y \in \mathfrak{m}$.*

$$Ric(X, Y) = -\frac{1}{4} tr(2P_X^* P_Y + T_X T_Y) - \frac{1}{2} \Phi(X, Y) + tr(P_{U(X, Y)}).$$

Remark 2.2. *If there exists on \mathfrak{m} a non-degenerate $Ad L$ -invariant symmetric bilinear form β , then $tr(P_{U(X, Y)}) = 0$, for all $X, Y \in \mathfrak{m}$. Indeed, if such a bilinear form exists, $tr P_a = 0$, for every $a \in \mathfrak{m}$. Let $\{w_i\}_i$ and $\{w'_i\}_i$ be bases of \mathfrak{m} dual with respect to β , i.e., $\beta(w_i, w'_j) = \delta_{ij}$. Then, for every $a \in \mathfrak{m}$,*

$$\beta(P_a w_i, w'_i) = \beta([a, w_i]_{\mathfrak{m}}, w'_i) = -\beta(w_i, [a, w'_i]_{\mathfrak{m}}) = -\beta(P_a w'_i, w_i).$$

Hence, $tr(P_a) = 0$. Also, if the metric g_M on M is naturally reductive, then $P_{U(X, Y)} = 0$, for all $X, Y \in \mathfrak{m}$, since, in this case, U is identically zero.

◇

Definition 2.3. *Let β be a non-degenerate $Ad G$ -invariant symmetric bilinear form on \mathfrak{g} . Let U be an $Ad L$ -invariant vector subspace of \mathfrak{g} such that the restriction of β to U is non-degenerate. The Casimir operator of U with respect to β is the operator*

$$C_U = \sum_i ad_{u_i} ad_{u'_i},$$

where $\{u_i\}_i$ and $\{u'_i\}_i$ are bases of U which are dual with respect to β , i.e., $\beta(u_i, u'_j) = \delta_{ij}$.

A Casimir operator is independent of the choice of dual basis. Moreover, it is an $Ad L$ -invariant linear map and thus it is scalar on any irreducible $Ad L$ -module. In particular, if \mathfrak{g} is simple, the only non-degenerate $Ad G$ -invariant symmetric bilinear map on \mathfrak{g} , up to scalar factor, is the Killing form Φ and $C_{\mathfrak{g}} = Id$.

Definition 2.4. Let U, V be $Ad L$ -invariant vector subspaces of \mathfrak{g} . We define a bilinear map $Q_{UV} : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathbb{R}$ by

$$Q_{UV}(X, Y) = tr([X, [Y, \cdot]_V]_U), \quad X, Y \in \mathfrak{m},$$

where the subscripts U and V denote the projections onto U and V , respectively.

Lemma 2.5. Let β be a non-degenerate $Ad G$ -invariant symmetric bilinear form on \mathfrak{g} . Let U, V be $Ad L$ -invariant vector subspaces of \mathfrak{g} such that the restrictions of β to U and V are both non-degenerate. Then

(i) $Q_{UV} = Q_{VU}$ and Q_{UV} is an $Ad L$ -invariant symmetric bilinear map. Hence, if $W \subset \mathfrak{g}$ is any irreducible $Ad L$ -submodule, then $Q_{UV}|_{W \times W}$ is a multiple of $\beta|_{W \times W}$.

(ii) if $ad_X U \subset V$ or $ad_Y U \subset V$, then $Q_{UV}(X, Y) = \beta(C_U X, Y) = \beta(X, C_U Y)$, for $X, Y \in \mathfrak{m}$, where C_U and C_V are the Casimir operators of U and V , respectively, with respect to β ;

(iii) if $ad_X V \perp U$ or $ad_Y V \perp U$, then $Q_{UV}(X, Y) = 0$, for $X, Y \in \mathfrak{m}$;

(iv) if $ad_X ad_Y U \perp U$ or $ad_Y ad_X U \perp U$, then $Q_{UV}(X, Y) = 0$, for $X, Y \in \mathfrak{m}$.

Proof: Since β is non-degenerate and $\beta|_{U \times U}, \beta|_{V \times V}$ are non-degenerate, we consider the orthogonal complements U^\perp and V^\perp of U and V , respectively, in \mathfrak{g} with respect to β . Also, we consider bases $\{w_i\}_i$ and $\{w'_i\}_i$ of U which are dual with respect to β . Moreover, since β is $Ad L$ -invariant, it is associative. Let $X, Y \in \mathfrak{m}$ and $g \in L$.

$$\begin{aligned} \beta([X, [Y, w_i]_V]_U, w'_i) &= \beta([X, [Y, w_i]_V], w'_i) \\ &= -\beta([Y, w_i]_V, [X, w'_i]) \\ &= -\beta([Y, w_i], [X, w'_i]_V) \\ &= \beta(w_i, [Y, [X, w'_i]_V]) \\ &= \beta(w_i, [Y, [X, w'_i]_V]_U). \end{aligned}$$

Therefore, $tr([X, [Y, \cdot]_V]_U) = tr([Y, [X, \cdot]_V]_U)$ and thus $Q_{UV}(X, Y) = Q_{UV}(Y, X)$. So Q_{UV} is symmetric. To show the $Ad L$ -invariance of Q_{UV} we note that since V and V^\perp are $Ad L$ -invariant subspaces and $\mathfrak{g} = V \oplus V^\perp$, the projections on V and V^\perp are also $Ad L$ -invariant linear maps.

$$\begin{aligned}
\beta([Ad_g X, [Ad_g Y, w_i]_V]_U, w'_i) &= \beta([Ad_g X, [Ad_g Y, w_i]_V], w'_i) \\
&= \beta(Ad_{g^{-1}}[Ad_g X, [Ad_g Y, w_i]_V], Ad_{g^{-1}} w'_i) \\
&= \beta([X, Ad_{g^{-1}}[Ad_g Y, w_i]_V], Ad_{g^{-1}} w'_i) \\
&= \beta([X, [Y, Ad_{g^{-1}} w_i]_V], Ad_{g^{-1}} w'_i) \\
&= \beta([X, [Y, Ad_{g^{-1}} w_i]_V]_U, Ad_{g^{-1}} w'_i).
\end{aligned}$$

Since β is $Ad G$ -invariant, if $\{w_i\}_i$ and $\{w'_i\}_i$ are dual bases of U with respect to β , then $\{Ad_{g^{-1}} w_i\}_i$ and $\{Ad_{g^{-1}} w'_i\}_i$ are still dual bases as well. So by the above we conclude that $tr([Ad_g X, [Ad_g Y, \cdot]_V]_U) = tr([X, [Y, \cdot]_V]_U)$ and thus Q_{UV} is $Ad L$ -invariant.

Let $Z \in \mathfrak{m}$ and set $A_Z = (ad_Z |_U)_V$ and $B_Z = (ad_Z |_V)_U$. We have

$$Q_{VU}(X, Y) = tr(A_X B_Y) = tr(B_Y A_X) = Q_{UV}(Y, X).$$

Hence, by symmetry of Q_{UV} , we conclude that $Q_{VU}(X, Y) = Q_{UV}(Y, X) = Q_{UV}(X, Y)$, for every $X, Y \in \mathfrak{m}$. Therefore, $Q_{UV} = Q_{VU}$.

To show (ii) let $C_U = \sum_i ad_{w_i} ad_{w'_i}$ be the Casimir operator of U with respect to β . Since $Q_{UV} = Q_{VU}$ it suffices to suppose that $ad_Y U \subset V$. If $ad_Y U \subset V$, then

$$Q_{UV}(X, Y) = tr([X, [Y, \cdot]_U]) = tr(ad_X ad_Y |_U).$$

Since $\beta([X, [Y, w_i]]_U, w'_i) = \beta([X, [Y, w_i]], w'_i) = \beta(Y, [w_i, [w'_i, X]])$, we have

$$Q_{UV}(X, Y) = \sum_i \beta(Y, [w_i, [w'_i, X]]) = \beta(Y, C_U X).$$

By symmetry of Q_{UV} we also get $Q_{UV}(X, Y) = \beta(X, C_U Y)$.

If $ad_X V \perp U$, then, for every $w, w' \in U$, $\beta([X, [Y, w]_V], w') = 0$ and thus $Q_{UV}(X, Y) = 0$, for every $Y \in \mathfrak{m}$. By symmetry, the same conclusion holds if $ad_Y V \perp U$. This shows (iii).

Finally, to prove (iv), if $ad_X ad_Y U \perp U$, then, for every $w, w' \in U$, $\beta([X, [Y, w]], w') = 0$ and thus $\beta([X, [Y, w]_V]_U, w') = 0$. Hence $Q_{UV}(X, Y) = 0$. If $ad_Y ad_X U \perp U$, then $Q_{UV}(X, Y) = 0$ by symmetry.

□

The following theorem describes the Ricci curvature of an invariant metric on $M = G/L$, when the Lie algebra admits a non-degenerate $Ad G$ -invariant symmetric bilinear form.

Theorem 2.6. *Let $M = G/L$ be a homogeneous manifold and let β be a non-degenerate $Ad G$ -invariant symmetric bilinear form on the Lie algebra \mathfrak{g} . Let g_M be the G -invariant pseudo-Riemannian metric on M induced by the scalar product of the form*

$$(2.9) \quad \langle \cdot, \cdot \rangle = \oplus_{j=1}^m \nu_j \beta|_{\mathfrak{m}_j \times \mathfrak{m}_j}, \quad \nu_j > 0,$$

where $\mathfrak{m} = \mathfrak{m}_1 \oplus \dots \oplus \mathfrak{m}_m$ is a β -orthogonal decomposition of \mathfrak{m} . For $X \in \mathfrak{m}_a$, $Y \in \mathfrak{m}_b$, the Ricci curvature of g_M is given by

$$Ric(X, Y) = \frac{1}{2} \sum_{j,k=1}^m \left(\frac{\nu_k}{\nu_j} - \frac{\nu_a \nu_b}{2\nu_k \nu_j} \right) Q_{\mathfrak{m}_j \mathfrak{m}_k}(X, Y) - \frac{1}{2} \Phi(X, Y).$$

Proof: First we note that $\beta|_{\mathfrak{m}_j \times \mathfrak{m}_j}$ is in fact non-degenerate. Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. We apply the formula given in Lemma 2.1. According to Remark 2.2, we have $tr(P_{U(X,Y)}) = 0$. Let $j = 1, \dots, m$ and let $\{w_i\}_i$ and $\{w'_i\}_i$ be dual bases for \mathfrak{m}_j with respect to β .

$$\begin{aligned} \langle T_X T_Y w_i, w'_i \rangle &= \langle X, [T_Y w_i, w'_i]_{\mathfrak{m}} \rangle \\ &= \nu_a \beta(X, [T_Y w_i, w'_i]) \\ &= -\nu_a \beta(T_Y w_i, [X, w'_i]) \\ &= -\nu_a \sum_{k=1}^m \beta(T_Y w_i, [X, w'_i]_{\mathfrak{m}_k}) \\ &= -\nu_a \sum_{k=1}^m \nu_k^{-1} \langle T_Y w_i, [X, w'_i]_{\mathfrak{m}_k} \rangle \\ &= -\nu_a \sum_{k=1}^m \nu_k^{-1} \langle Y, [w_i, [X, w'_i]_{\mathfrak{m}_k}]_{\mathfrak{m}} \rangle \\ &= -\nu_a \nu_b \sum_{k=1}^m \nu_k^{-1} \beta([Y, w_i], [X, w'_i]_{\mathfrak{m}_k}) \\ &= -\nu_a \nu_b \sum_{k=1}^m \nu_k^{-1} \beta([Y, w_i]_{\mathfrak{m}_k}, [X, w'_i]) \\ &= \nu_a \nu_b \sum_{k=1}^m \nu_k^{-1} \beta(w'_i, [X, [Y, w_i]_{\mathfrak{m}_k}]) \\ &= \nu_a \nu_b \sum_{k=1}^m \nu_k^{-1} \beta(w'_i, [X, [Y, w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) \\ &= \nu_a \nu_b \sum_{k=1}^m (\nu_k \nu_j)^{-1} \langle w'_i, [X, [Y, w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j} \rangle. \end{aligned}$$

This implies that

$$tr(T_X T_Y|_{\mathfrak{m}_j}) = \nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k \nu_j} tr([X, [Y, \cdot]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) = \nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k \nu_j} Q_{\mathfrak{m}_j \mathfrak{m}_k}(X, Y)$$

and thus $tr(T_X T_Y) = \nu_a \nu_b \sum_{j,k=1}^m \frac{1}{\nu_k \nu_j} Q_{\mathfrak{m}_j \mathfrak{m}_k}(X, Y)$.

$$\begin{aligned} \langle P_X^* P_Y w_i, w'_i \rangle &= \langle P_Y w_i, P_X w'_i \rangle \\ &= \sum_{k=1}^m \langle [Y, w_i]_{\mathfrak{m}_k}, [X, w'_i]_{\mathfrak{m}_k} \rangle \\ &= \sum_{k=1}^m \nu_k \beta([Y, w_i]_{\mathfrak{m}_k}, [X, w'_i]) \\ &= -\sum_{k=1}^m \nu_k \beta(w'_i, [X, [Y, w_i]_{\mathfrak{m}_k}]) \\ &= -\sum_{k=1}^m \nu_k \beta(w'_i, [X, [Y, w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) \\ &= -\sum_{k=1}^m \nu_k \nu_j^{-1} \langle w'_i, [X, [Y, w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j} \rangle. \end{aligned}$$

Then

$$\text{tr}(P_X^* P_Y |_{\mathfrak{m}_j}) = - \sum_{k=1}^m \frac{\nu_k}{\nu_j} \text{tr}([X, [Y, \cdot]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) = - \sum_{k=1}^m \frac{\nu_k}{\nu_j} Q_{\mathfrak{m}_j \mathfrak{m}_k}(X, Y)$$

and thus we get $\text{tr}(P_X^* P_Y) = - \sum_{j,k=1}^m \frac{\nu_k}{\nu_j} Q_{\mathfrak{m}_j \mathfrak{m}_k}(X, Y)$.

By using Lemma 2.1 we finally obtain the required expression for $\text{Ric}(X, Y)$.

□

We recall that a metric g_M is said to be *normal* if it is the restriction of a non-degenerate $\text{Ad } L$ -invariant symmetric bilinear form on \mathfrak{m} . The formula below for the Ricci curvature of a normal metric was first found by Wang and Ziller in [25], and can be deduced from Theorem 2.6. From Corollary 2.7, it is clear that a necessary and sufficient condition for a normal metric to be Einstein is that the Casimir operator of \mathfrak{l} is scalar on the isotropy space \mathfrak{m} . For instance, this condition holds if \mathfrak{m} is irreducible. simply-connected non-strongly isotropy irreducible homogeneous spaces which admit a normal Einstein metric were classified by Wang and Ziller in [25], when G is a compact connected simple group. Also, more generally, simply-connected compact standard homogeneous manifolds were studied by E.D. Rodionov in [22].

Corollary 2.7. *Let β be a non-degenerate $\text{Ad } G$ -invariant symmetric bilinear form on \mathfrak{g} and g_M the normal metric on M defined by the restriction of β to \mathfrak{m} . The Ricci curvature of g_M is given by $\text{Ric}(\mathfrak{m}_a, \mathfrak{m}_b) = 0$, if $a \neq b$, and for $X \in \mathfrak{m}_a$,*

$$\text{Ric}(X, X) = -\frac{1}{4}\Phi(X, X) - \frac{1}{2}\beta(C_{\mathfrak{l}}X, X),$$

where $C_{\mathfrak{l}}$ is the Casimir operator of \mathfrak{l} with respect to β .

Proof: Let $C_{\mathfrak{g}}$, $C_{\mathfrak{l}}$ and $C_{\mathfrak{m}}$ be the Casimir operators of \mathfrak{g} , \mathfrak{l} and \mathfrak{m} with respect to β . We remark that the Killing form of \mathfrak{g} may not be non-degenerate. Since g_M is defined by the restriction of β , in Theorem 2.6 we can take $\nu_1 = \dots = \nu_m = 1$ and obtain the following. Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$.

$$\begin{aligned} \text{Ric}(X, Y) &= \frac{1}{4} \sum_{j,k=1}^m Q_{\mathfrak{m}_j \mathfrak{m}_k}(X, Y) - \frac{1}{2}\Phi(X, Y) \\ &= \frac{1}{4}Q_{\mathfrak{m}\mathfrak{m}}(X, Y) - \frac{1}{2}\Phi(X, Y) \\ &= \frac{1}{4}Q_{\mathfrak{m}\mathfrak{g}}(X, Y) - \frac{1}{4}Q_{\mathfrak{m}\mathfrak{l}}(X, Y) - \frac{1}{2}\Phi(X, Y) \\ &= \frac{1}{4}\beta(C_{\mathfrak{m}}X, Y) - \frac{1}{4}\beta(C_{\mathfrak{l}}X, Y) - \frac{1}{2}\Phi(X, Y) \\ &= -\frac{1}{4}\Phi(X, Y) - \frac{1}{2}\beta(C_{\mathfrak{l}}X, Y). \end{aligned}$$

Since $C_{\mathfrak{l}}(\mathfrak{m}_a) \subset \mathfrak{m}_a$, it is clear that $\text{Ric}(X, Y) = 0$ if $a \neq b$ and Ric is well determined by elements $\text{Ric}(X, X)$ with $X \in \mathfrak{m}_a$.

□

We obtain a similar formula to that of Corollary 2.7, in the case when the submodules $\mathfrak{m}_1, \dots, \mathfrak{m}_m$ pairwise commute. The proof is similar.

Corollary 2.8. *Let β be a non-degenerate $\text{Ad } G$ -invariant symmetric bilinear form on \mathfrak{g} and g_M on M defined by (2.9). Suppose that $[\mathfrak{m}_a, \mathfrak{m}_b] = 0$, if $a \neq b$. Then $\text{Ric}(\mathfrak{m}_a, \mathfrak{m}_b) = 0$, for $a \neq b$, and for $X \in \mathfrak{m}_a$,*

$$\text{Ric}(X, X) = -\frac{1}{4}\Phi(X, X) - \frac{1}{2}\beta(C_{\mathfrak{l}}X, X),$$

where $C_{\mathfrak{l}}$ is the Casimir operator of \mathfrak{l} with respect to the β .

3. THE RICCI CURVATURE OF AN ADAPTED METRIC

In this section we obtain the Ricci curvature of an adapted metric g_M on the total space of a homogeneous fibration as in (1.1). Let

$$M = G/L \rightarrow G/K = N,$$

with fiber $F = K/L$, for a compact connected semisimple Lie group G and

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p} \oplus \mathfrak{n}$$

an associated reductive decomposition. We use the notation convention from Section 1. We recall that an adapted metric g_M on M is induced by an $Ad L$ -invariant Euclidean product $g_{\mathfrak{m}}$ given by (1.4), i.e.,

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

All the Casimir operators $C_{\mathfrak{k}}$, $C_{\mathfrak{p}_a}$ and $C_{\mathfrak{n}_j}$ are with respect to the Killing form Φ (see Definition 1.5). Since $\Phi(C_{\mathfrak{k}}\cdot, \cdot)$ and $\Phi(C_{\mathfrak{n}_j}\cdot, \cdot)$ are $Ad L$ -invariant symmetric bilinear maps and \mathfrak{p}_a is $Ad L$ -irreducible, there are constants γ_a and $c_{\mathfrak{n}_j, a}$ such that

$$(3.1) \quad \Phi(C_{\mathfrak{k}}\cdot, \cdot) \big|_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a \Phi \big|_{\mathfrak{p}_a \times \mathfrak{p}_a}$$

$$(3.2) \quad \Phi(C_{\mathfrak{n}_j}\cdot, \cdot) \big|_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{\mathfrak{n}_j, a} \Phi \big|_{\mathfrak{p}_a \times \mathfrak{p}_a}.$$

In the following Lemma we use the bilinear form Q_{UV} from Definition 2.4.

Lemma 3.1. *Let $X \in \mathfrak{p}$ and $Y \in \mathfrak{m}$.*

- (i) $Q_{\mathfrak{n}_j \mathfrak{p}_a}(X, Y) = Q_{\mathfrak{p}_a \mathfrak{n}_j}(X, Y) = 0$;
- (ii) $Q_{\mathfrak{n}_i \mathfrak{n}_j}(X, Y) = 0$, if $i \neq j$;
- (iii) $Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, Y) = \Phi(C_{\mathfrak{n}_j}X, Y)$.

Let $X' \in \mathfrak{n}_k$ and $Y' \in \mathfrak{m}$.

- (iv) $Q_{\mathfrak{n}_j \mathfrak{p}_a}(X', Y') = Q_{\mathfrak{p}_a \mathfrak{n}_j}(X', Y') = 0$, if $j \neq k$;
- (v) $Q_{\mathfrak{p}_a \mathfrak{n}_k}(X', Y') = Q_{\mathfrak{n}_k \mathfrak{p}_a}(X', Y') = \Phi(C_{\mathfrak{p}_a}X', Y')$;
- (vi) $Q_{\mathfrak{p}_b \mathfrak{p}_a}(X', Y') = 0$.

Proof: Let $X \in \mathfrak{p}$ and $Y \in \mathfrak{m}$. Since $ad_X \mathfrak{p} \subset \mathfrak{k} \perp \mathfrak{n}$ we have $Q_{\mathfrak{n}_j \mathfrak{p}_a}(X, Y) = 0$, from Lemma 2.5. From Lemma 2.5, $Q_{\mathfrak{p}_a \mathfrak{n}_j}(X, Y) = Q_{\mathfrak{n}_j \mathfrak{p}_a}(X, Y) = 0$. As $ad_X \mathfrak{n}_j \subset \mathfrak{n}_j$, we have $Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, Y) = \Phi(C_{\mathfrak{n}_j}X, Y)$. Moreover, since $\mathfrak{n}_j \perp \mathfrak{n}_i$, for every $i \neq j$, we also conclude that $Q_{\mathfrak{n}_i \mathfrak{n}_j}(X, Y) = 0$, if $i \neq j$.

Let $X' \in \mathfrak{n}_k$ and $Y' \in \mathfrak{m}$. We have $ad_{X'} \mathfrak{p}_a \subset \mathfrak{n}_k \perp \mathfrak{p}, \mathfrak{n}_j$, for $j \neq k$. Thus, $Q_{\mathfrak{n}_j \mathfrak{p}_a}(X', Y') = 0$, if $j \neq k$ and $Q_{\mathfrak{p}_j \mathfrak{p}_a}(X', Y') = 0$, from Lemma 2.5. Also from $ad_{X'} \mathfrak{p}_a \subset \mathfrak{n}_k$ we deduce that $Q_{\mathfrak{p}_a \mathfrak{n}_k}(X', Y') = \Phi(C_{\mathfrak{p}_a}X', Y')$. From Lemma 2.5, we also obtain $Q_{\mathfrak{p}_a \mathfrak{n}_j}(X', Y') = Q_{\mathfrak{n}_j \mathfrak{p}_a}(X', Y') = 0$, for $j \neq k$ and $Q_{\mathfrak{n}_k \mathfrak{p}_a}(X', Y') = Q_{\mathfrak{p}_a \mathfrak{n}_k}(X', Y') = \Phi(C_{\mathfrak{p}_a}X', Y')$, for $j = k$. \square

In the remaining of this section, we obtain formulae for the Ricci curvature of an adapted metric of the fibration $M = G/L \rightarrow G/K = N$ in the vertical, horizontal

and $\mathfrak{p} \times \mathfrak{n}$ directions, by using Lemma 3.1 and the formula for the Ricci curvature from Theorem 2.6.

3.1. The Ricci Curvature of an Adapted metric in the Vertical Direction.

Lemma 3.2. *Let g_F be the K -invariant metric on the fiber $F = K/L$ determined by the $Ad L$ -invariant Euclidean product $g_{\mathfrak{p}} = \oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}$. The Ricci curvature of g_F is given by $Ric^F = \oplus_{a=1}^s q_a B_{\mathfrak{p}_a}$, where*

$$(3.3) \quad q_a = \frac{1}{2} \sum_{b,c=1}^s \left(\frac{\lambda_a^2}{2\lambda_c\lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2}.$$

The constants q_a^{cb} and γ_a are such that

$$(3.4) \quad \Phi_{\mathfrak{t}}|_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a}$$

$$(3.5) \quad Q_{\mathfrak{p}_b \mathfrak{p}_c}|_{\mathfrak{p}_a \times \mathfrak{p}_a} = q_a^{cb} \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a}.$$

In particular, $Ric^F(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$.

Proof: Since $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are pairwise inequivalent irreducible $Ad L$ -submodules and the Ricci curvature of g_F , Ric^F , is an $Ad L$ -invariant symmetric bilinear form, we may write $Ric^F = \oplus_{a=1}^s q_a B_{\mathfrak{p}_a}$, for some constants q_1, \dots, q_s . In particular, we have $Ric^F(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$. By Theorem 2.6, the Ricci curvature of g_F is

$$Ric^F(X, X) = \frac{1}{2} \sum_{b,c=1}^s \left(\frac{\lambda_b}{\lambda_c} - \frac{\lambda_a^2}{2\lambda_c\lambda_b} \right) Q_{\mathfrak{p}_c \mathfrak{p}_b}(X, X) - \frac{1}{2} \Phi_{\mathfrak{t}}(X, X).$$

By Lemma 2.5 the maps $Q_{\mathfrak{p}_c \mathfrak{p}_b}$ are $Ad L$ -invariant symmetric bilinear maps. Since \mathfrak{p}_a is $Ad L$ -irreducible, there are constants q_a^{cb} as defined by (3.5). Similarly, there is a constant γ_a as defined in (3.4). By the expression above for Ric^F , we must have

$$q_a = \frac{1}{2} \sum_{b,c=1}^s \left(\frac{\lambda_a^2}{2\lambda_c\lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2}$$

and the result follows from this.

□

Proposition 3.3. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

We have $Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$. For $X \in \mathfrak{p}_a$,

$$Ric(X, X) = \left(q_a + \frac{\lambda_a^2}{4} \sum_{j=1}^n \frac{c_{\mathfrak{n}_j, a}}{\mu_j^2} \right) B(X, X),$$

where, for $j = 1, \dots, n$, the constants $c_{\mathfrak{n}_j, a}$ are such that

$$(3.6) \quad \Phi(C_{\mathfrak{n}_j} \cdot, \cdot)|_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{\mathfrak{n}_j, a} \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a}$$

and $C_{\mathfrak{n}_j}$ is the Casimir operator of \mathfrak{n}_j with respect to Φ . The constant q_a is such that

$$q_a = \frac{1}{2} \sum_{b,c=1}^s \left(\frac{\lambda_a^2}{2\lambda_c\lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2},$$

with q_a^{cb} and γ_a defined by

$$(3.7) \quad \Phi_{\mathfrak{k}}|_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a}$$

$$(3.8) \quad Q_{\mathfrak{p}_b \mathfrak{p}_c}|_{\mathfrak{p}_a \times \mathfrak{p}_a} = q_a^{cb} \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a}.$$

Proof: Since $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are pairwise inequivalent irreducible $Ad L$ -submodules and $Ric|_{\mathfrak{p} \times \mathfrak{p}}$ is an $Ad L$ -invariant symmetric bilinear form, we have that $Ric|_{\mathfrak{p} \times \mathfrak{p}}$ is diagonal, i.e.,

$$Ric|_{\mathfrak{p} \times \mathfrak{p}} = a_1 B_{\mathfrak{p}_1} \oplus \dots \oplus a_s B_{\mathfrak{p}_s},$$

for some constants a_1, \dots, a_s . In particular, we have $Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$. Hence, $Ric|_{\mathfrak{p} \times \mathfrak{p}}$ is determined by elements $Ric(X, X)$ with $X \in \mathfrak{p}_a$, $a = 1, \dots, s$.

By Lemma 3.1 we obtain that only $Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, X) = \Phi(C_{\mathfrak{n}_j} X, X)$ and $Q_{\mathfrak{p}_b \mathfrak{p}_c}(X, X)$ is non-zero. Therefore, by Theorem 2.6 we obtain that

$$Ric(X, X) =$$

$$\frac{1}{2} \sum_{j,k=1}^s \left(\frac{\lambda_k}{\lambda_j} - \frac{\lambda_a^2}{2\lambda_j\lambda_k} \right) Q_{\mathfrak{p}_j \mathfrak{p}_k}(X, X) + \frac{1}{2} \sum_{j=1}^n \left(1 - \frac{\lambda_a^2}{2\mu_j^2} \right) Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, X) - \frac{1}{2} \Phi(X, X).$$

We have $\sum_{j=1}^m Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, X) = \sum_{j=1}^m \Phi(C_{\mathfrak{n}_j} X, X) = \Phi(X, X) - \Phi_{\mathfrak{k}}(X, X)$.

Hence we can rewrite $Ric(X, X)$ as follows:

$$\underbrace{\frac{1}{2} \sum_{a,b=1}^s \left(\frac{\lambda_b}{\lambda_c} - \frac{\lambda_a^2}{2\lambda_c\lambda_b} \right) Q_{\mathfrak{p}_c \mathfrak{p}_b}(X, X) - \frac{1}{2} \Phi_{\mathfrak{k}}(X, X) - \frac{1}{2} \sum_{j=1}^n \frac{\lambda_a^2}{2\mu_j^2} \Phi(C_{\mathfrak{n}_j} X, X)}_{(1)}.$$

As we saw in the proof of Lemma 3.2, the summand (1) is just $Ric^F(X, X) = q_a B(X, X)$. Furthermore, since $\Phi(C_{\mathfrak{n}_j} \cdot, \cdot) = Q_{\mathfrak{n}_j \mathfrak{n}_j}$ there are constants $c_{\mathfrak{n}_j, a}$, defined in (3.6).

Therefore,

$$Ric(X, X) = \left(q_a + \frac{1}{4} \sum_{j=1}^n \frac{\lambda_a^2}{\mu_j^2} c_{\mathfrak{n}_j, a} \right) B(X, X).$$

□

3.2. The Ricci Curvature of an Adapted metric in the Horizontal Direction.

Lemma 3.4. *Let g_N be the G -invariant metric on $N = G/K$ determined by the $Ad L$ -invariant Euclidean product $g_{\mathfrak{n}} = \oplus_{j=1}^n \mu_j B_{\mathfrak{n}_j}$. The Ricci curvature of g_N is given by $Ric^N = \oplus_{k=1}^n r_k B_{\mathfrak{n}_k}$, where*

$$(3.9) \quad r_k = \frac{1}{2} \sum_{j,i=1}^n \left(\frac{\mu_a^2}{2\mu_i \mu_j} - \frac{\mu_i}{\mu_j} \right) r_k^{ji} + \frac{1}{2}$$

and the constants r_k^{ji} are such that

$$(3.10) \quad Q_{\mathfrak{n}_j \mathfrak{n}_i} |_{\mathfrak{n}_k \times \mathfrak{n}_k} = r_k^{ji} \Phi |_{\mathfrak{n}_k \times \mathfrak{n}_k}.$$

In particular, $Ric^N(\mathfrak{n}_k, \mathfrak{n}_j) = 0$, if $k \neq j$.

Proof: The proof is similar to the proof of Lemma 3.2 and it can be found in detail in [3].

□

Proposition 3.5. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

We have $Ric(\mathfrak{n}_k, \mathfrak{n}_j) = 0$, if $j \neq k$. For $X \in \mathfrak{n}_k$,

$$Ric(X, X) = -\frac{1}{2\mu_k} \sum_{a=1}^s \lambda_a B(C_{\mathfrak{p}_a} X, X) + r_k B(X, X),$$

where $C_{\mathfrak{p}_a}$ is the Casimir operator of \mathfrak{p}_a with respect to Φ ,

$$r_k = \frac{1}{2} \sum_{j,i=1}^n \left(\frac{\mu_a^2}{2\mu_i \mu_j} - \frac{\mu_i}{\mu_j} \right) r_k^{ji} + \frac{1}{2}$$

and the constants r_k^{ji} are such that

$$Q_{\mathfrak{n}_j \mathfrak{n}_i} |_{\mathfrak{n}_k \times \mathfrak{n}_k} = r_k^{ji} \Phi |_{\mathfrak{n}_k \times \mathfrak{n}_k}.$$

Proof: Let $X \in \mathfrak{n}_k$ and $Y \in \mathfrak{n}_{k'}$. By Lemma 3.1 we have $Q_{\mathfrak{p}_a \mathfrak{p}_b}(X, Y) = 0$, for every $a, b = 1, \dots, s$. Also, $Q_{\mathfrak{n}_j \mathfrak{p}_a}(X, Y) = Q_{\mathfrak{p}_a \mathfrak{n}_j}(X, Y) = 0$, if $j \neq k, k'$. Therefore, it follows from Theorem 2.6 and Lemma 3.4 that, if $k \neq k'$, then

$$Ric(X, Y) = \frac{1}{2} \sum_{j,i=1}^n \left(\frac{\mu_i}{\mu_j} - \frac{\mu_k \mu_{k'}}{2\mu_i \mu_j} \right) Q_{\mathfrak{n}_j \mathfrak{n}_i}(X, Y) - \frac{1}{2} \Phi(X, Y) = Ric^N(X, Y) = 0.$$

Hence, $Ric |_{\mathfrak{n} \times \mathfrak{n}}$ is determined by elements $Ric(X, X)$ with $X \in \mathfrak{n}_k$, $k = 1, \dots, n$. For $X \in \mathfrak{n}_k$, by Theorem 2.6, we get

$$Ric(X, X) =$$

$$\frac{1}{2} \sum_{k=1}^s \left(\frac{\mu_k}{\lambda_a} - \frac{\mu_k^2}{2\mu_k \lambda_a} \right) Q_{\mathfrak{p}_a \mathfrak{n}_k}(X, X) + \frac{1}{2} \sum_{a=1}^s \left(\frac{\lambda_a}{\mu_k} - \frac{\mu_k^2}{2\mu_k \lambda_a} \right) Q_{\mathfrak{n}_k \mathfrak{p}_a}(X, X) + Ric^N(X, X).$$

From Lemma 3.1, we know that $Q_{\mathfrak{n}_k \mathfrak{p}_a}(X, X) = Q_{\mathfrak{p}_a \mathfrak{n}_k}(X, X) = \Phi(C_{\mathfrak{p}_a} X, X)$. Hence, we simplify the expression above obtaining

$$Ric(X, X) = \frac{1}{2} \sum_{a=1}^s \frac{\lambda_a}{\mu_k} \Phi(C_{\mathfrak{p}_a} X, X) + Ric^N(X, X).$$

Finally, by using Lemma 3.4 we have $Ric^N(X, X) = r_k B(X, X)$ and this concludes the proof.

□

3.3. The Ricci Curvature of an Adapted metric in the Mixed Direction $\mathfrak{p} \times \mathfrak{n}$.

Proposition 3.6. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

For $X \in \mathfrak{p}_a, Y \in \mathfrak{n}_k$,

$$Ric(X, Y) = \frac{\lambda_a \mu_k}{4} \sum_{j=1}^n \frac{B(C_{\mathfrak{n}_j} X, Y)}{\mu_j^2},$$

where $C_{\mathfrak{n}_j}$ is the Casimir operator of \mathfrak{n}_j with respect to Φ .

Proof: For $X \in \mathfrak{p}$ we know from Lemma 3.1 that $Q_{\mathfrak{n}_j \mathfrak{p}_a}(X, Y) = Q_{\mathfrak{p}_a \mathfrak{n}_j}(X, Y) = 0$ and $Q_{\mathfrak{n}_i \mathfrak{n}_j}(X, Y) = 0$, if $i \neq j$, whereas $Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, Y) = \Phi(C_{\mathfrak{n}_j} X, Y)$. Moreover, for $Y \in \mathfrak{n}_k$, since $ad_X ad_Y \mathfrak{p} \subset \mathfrak{n}_k \perp \mathfrak{p}$, we also obtain from Lemma 3.1 that $Q_{\mathfrak{p}_b \mathfrak{p}_c}(X, Y) = 0$. Therefore, only $Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, Y) = \Phi(C_{\mathfrak{n}_j} X, Y)$, may not be zero. Furthermore, $\Phi(X, Y) = 0$. Hence, from Theorem 2.6 we get

$$Ric(X, Y) = \frac{1}{2} \sum_{j=1}^n \left(1 - \frac{\lambda_a \mu_k}{2\mu_j^2} \right) Q_{\mathfrak{n}_j \mathfrak{n}_j}(X, Y) = \frac{1}{2} \sum_{j=1}^n \left(1 - \frac{\lambda_a \mu_k}{2\mu_j^2} \right) \Phi(C_{\mathfrak{n}_j} X, Y).$$

On the other hand,

$$\sum_{j=1}^n \Phi(C_{\mathfrak{n}_j} X, Y) = \Phi(C_{\mathfrak{n}} X, Y) = \Phi(X, Y) - \Phi(C_{\mathfrak{k}} X, Y) = 0,$$

since $C_{\mathfrak{k}} \mathfrak{p} \subset \mathfrak{k} \perp \mathfrak{n}$. Therefore,

$$Ric(X, Y) = -\frac{\lambda_a \mu_k}{4} \sum_{j=1}^n \frac{\Phi(C_{\mathfrak{n}_j} X, Y)}{\mu_j^2}.$$

□

3.4. Necessary Conditions for the Existence of an Adapted Einstein Metric. From the expressions obtained previously for the Ricci curvature in the horizontal direction and in the direction of $\mathfrak{p} \times \mathfrak{n}$ we obtain two necessary conditions for the existence of an adapted Einstein metric on M .

Corollary 3.7. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

If g_M is Einstein, then the operator $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}$ is scalar on each \mathfrak{n}_j .

Proof: Let g_M be an adapted metric as defined in (1.4). If g_M is Einstein with Einstein constant E , then, $Ric|_{\mathfrak{n} \times \mathfrak{n}} = E g_{\mathfrak{n}}$ and thus $Ric|_{\mathfrak{n} \times \mathfrak{n}}$ is $Ad K$ -invariant. Therefore, by Proposition 3.5, we conclude that $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}|_{\mathfrak{n}}$ must be $Ad K$ -invariant. Hence, $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}|_{\mathfrak{n}_k}$ is scalar, by irreducibility of \mathfrak{n}_k . \square

Corollary 3.8. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

The orthogonality condition $Ric(\mathfrak{p}, \mathfrak{n}) = 0$ holds if and only if

$$(3.11) \quad \sum_{j=1}^n \frac{1}{\mu_j^2} C_{\mathfrak{n}_j}(\mathfrak{p}) \subset \mathfrak{k}.$$

Moreover, if g_M is Einstein, then (3.11) holds.

Proof: From Proposition 3.6, we obtain that $Ric(\mathfrak{p}, \mathfrak{n}) = 0$ if and only if, for every $X \in \mathfrak{p}_a$ and $Y \in \mathfrak{n}_b$,

$$\Phi \left(\sum_{j=1}^n \frac{C_{\mathfrak{n}_j}}{\mu_j^2} X, Y \right) = 0.$$

This holds if only if $\sum_{j=1}^n \frac{C_{\mathfrak{n}_j}}{\mu_j^2} X \subset \mathfrak{k}$, for every $X \in \mathfrak{p}$.

If g_M is Einstein with Einstein constant E , then $Ric(\mathfrak{p}, \mathfrak{n}) = E g_{\mathfrak{m}}(\mathfrak{p}, \mathfrak{n}) = 0$. \square

The two previous Corollaries may be restated as in Theorem 1.2, which emphasizes the fact that the two necessary conditions obtained for existence of an Einstein adapted metric are just algebraic conditions on the Casimir operators of the submodules \mathfrak{p}_a and \mathfrak{n}_k .

Proof of Theorem 1.2: The assertions follow from Corollaries 3.7 and 3.8. In 3.8 we set $\nu_k = 1/\mu_k^2$. Hence, $\nu_1 = \dots = \nu_n$ occurs when g_N is the standard metric. Moreover, if $\nu_1 = \dots = \nu_n$, the inclusion in Theorem 3.8 is equivalent to $C_{\mathfrak{n}}(\mathfrak{p}) \subset \mathfrak{k}$, which always holds since $C_{\mathfrak{n}} = Id - C_{\mathfrak{k}}$ and $C_{\mathfrak{k}}$ maps \mathfrak{p} into \mathfrak{k} . So we obtain a condition on the $C_{\mathfrak{n}_j}$'s only when g_N is not standard. \square

4. EINSTEIN BINORMAL METRICS

In this Section we obtain the Ricci curvature of a binormal metric (see (1.6)) and conditions for such a metric to be Einstein. We prove Theorem 1.3 and the subsequent Corollaries stated in Section 1. As we shall see, the conditions for the existence of an Einstein binormal metric translate in very simple conditions on the Casimir operators of \mathfrak{k} , \mathfrak{l} and \mathfrak{p}_a . As in previous sections, we consider the homogeneous fibration $M = G/L \rightarrow G/K = N$ as in (1.1). We use the notation from Sections 1 and 3. We start by considering the case when the restriction g_F of the metric to the fiber $F = K/L$ is normal.

Proposition 4.1. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K$ defined by the Euclidean product*

$$g_M = \lambda B_{\mathfrak{p}} \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

The Ricci curvature of g_M is given by:

(i) *For $X \in \mathfrak{p}_a$,*

$$Ric(X, X) = \left(q_a + \frac{\lambda^2}{4} \sum_{j=1}^n \frac{c_{\mathfrak{n}_j, a}}{\mu_j^2} \right) B(X, X),$$

where $q_a = \frac{1}{2} (c_{\mathfrak{l}, a} + \frac{\gamma_a}{2})$, $c_{\mathfrak{l}, a}$ is the eigenvalue of the Casimir operator of \mathfrak{l} on \mathfrak{p}_a , γ_a and $c_{\mathfrak{n}_j, a}$ are given by (3.4) and (3.6), respectively.

(ii) *For $X \in \mathfrak{n}_k$,*

$$Ric(X, X) = -\frac{\lambda}{2\mu_k} B(C_{\mathfrak{p}} X, X) + r_k B(X, X),$$

where r_k is given by (3.9);

(iii) *For $X \in \mathfrak{p}_a$ and $Y \in \mathfrak{n}_k$,*

$$Ric(X, Y) = \frac{\lambda \mu_k}{4} \sum_{j=1}^n \frac{B(C_{\mathfrak{n}_j} X, Y)}{\mu_j^2};$$

(iv) *$Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$, and $Ric(\mathfrak{n}_i, \mathfrak{n}_j) = 0$, if $i \neq j$.*

Proof: (i) By Corollary 2.7, if g_F is a multiple of $B = -\Phi$, then we obtain that

$$Ric^F(X, X) = -\frac{1}{4} \Phi_{\mathfrak{k}}(X, X) - \frac{1}{2} \Phi(C_{\mathfrak{l}} X, X) = -\frac{1}{2} \left(\frac{\gamma_a}{2} + c_{\mathfrak{l}, a} \right) \Phi(X, X),$$

for $X \in \mathfrak{p}_a$, where $c_{\mathfrak{l}, a}$ is the eigenvalue of the Casimir operator of \mathfrak{l} with respect to Φ on \mathfrak{p}_a . Hence, we have

$$q_a = \frac{1}{2} \left(\frac{\gamma_a}{2} + c_{\mathfrak{l}, a} \right).$$

The result then follows from this and Proposition 3.3.

(ii) It follows directly from Proposition 3.5, by observing that $\sum_{a=1}^s C_{\mathfrak{p}_a} = C_{\mathfrak{p}}$.

(iii) The Ricci curvature in the direction $\mathfrak{p} \times \mathfrak{n}$ essentially remains unchanged; the expression given is just that of Proposition 3.6 after replacing $\lambda_1, \dots, \lambda_s$ by λ .

(iv) These orthogonality conditions are satisfied by any adapted metric on M and were shown to hold in Propositions 3.3 and 3.5.

□

Similarly, if the metric on the base is normal, then we obtain the following characterization:

Proposition 4.2. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus \mu B_{\mathfrak{n}}.$$

The Ricci curvature of g_M is given by:

(i) *For $X \in \mathfrak{p}_a$,*

$$\text{Ric}(X, X) = \left(q_a + \frac{\lambda_a^2}{4\mu^2}(1 - \gamma_a) \right) B(X, X),$$

where q_a and γ_a are given by (3.4) and (3.10), respectively.

(ii) *For $X \in \mathfrak{n}_k$,*

$$\text{Ric}(X, X) = -\frac{1}{2} \sum_{a=1}^s \frac{\lambda_a}{\mu} B(C_{\mathfrak{p}_a} X, X) + r_k B(X, X),$$

with $r_k = \frac{1}{2} (\frac{1}{2} + c_{\mathfrak{k},k})$, where $c_{\mathfrak{k},k}$ is the eigenvalue of the Casimir operator $C_{\mathfrak{k}}$ on \mathfrak{n}_k ;

(iii) *$\text{Ric}(\mathfrak{p}, \mathfrak{n}) = 0$;*

(iv) *$\text{Ric}(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$, and $\text{Ric}(\mathfrak{n}_i, \mathfrak{n}_j) = 0$, if $i \neq j$.*

Proof: (i) From the fact that $\gamma_a + \sum_{j=1}^n c_{\mathfrak{n}_j, a} = 1$, we obtain

$$\sum_{j=1}^n \frac{\lambda_a^2}{\mu^2} c_{\mathfrak{n}_j, a} = \frac{\lambda_a^2}{\mu^2} (1 - \gamma_a).$$

The required expression follows immediately from Proposition 3.3.

(ii) From Corollary 2.7 we obtain that $r_k = \frac{1}{2} (\frac{1}{2} + c_{\mathfrak{k},k})$, where r_k is as defined in Lemma 3.4. The expression then follows from Proposition 3.5.

(iii) By using the fact that $C_{\mathfrak{n}} = \sum_{j=1}^n C_{\mathfrak{n}_j}$, from Proposition 3.6 it follows that

$$\text{Ric}(X, Y) = \frac{\lambda_a}{4\mu} B(C_{\mathfrak{n}} X, Y),$$

for every $X \in \mathfrak{p}_a$ and $Y \in \mathfrak{n}_k$. Moreover, since $C_{\mathfrak{n}} = C_{\mathfrak{g}} - C_{\mathfrak{k}} = \text{Id} - C_{\mathfrak{k}}$ and $C_{\mathfrak{k}}(\mathfrak{p}) \subset \mathfrak{k}$, we have that $C_{\mathfrak{n}}(X) \in \mathfrak{k}$ is orthogonal to $Y \in \mathfrak{n}$ with respect to B . Hence, $\text{Ric}(X, Y) = 0$.

(iv) these orthogonality conditions are simply those in Propositions 3.3 and 3.5.

□

The Ricci curvature of a binormal metric follows immediately from Propositions 4.1 and 4.2:

Corollary 4.3. *Consider the homogeneous fibration $M = G/L \rightarrow G/K$ and the binormal metric g_M on M defined by the Euclidean product $g_{\mathfrak{m}} = \lambda B_{\mathfrak{p}} \oplus \mu B_{\mathfrak{n}}$.*

(i) *For every $X \in \mathfrak{p}_a$,*

$$\text{Ric}(X, X) = \left(q_a + \frac{\lambda^2}{4\mu^2}(1 - \gamma_a) \right) B(X, X),$$

where $q_a = \frac{1}{2} (\frac{\gamma_a}{2} + c_{\mathfrak{l},a})$, $c_{\mathfrak{l},a}$ is the eigenvalue of $C_{\mathfrak{l}}$ on \mathfrak{p}_a and γ_a is determined by

$$\Phi_{\mathfrak{k}}|_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a \Phi|_{\mathfrak{p}_a \times \mathfrak{p}_a};$$

(ii) For every $Y \in \mathfrak{n}_j$,

$$\text{Ric}(Y, Y) = -\frac{\lambda}{2\mu} B(C_{\mathfrak{p}} Y, Y) + r_j B(Y, Y),$$

where $r_j = \frac{1}{2} \left(\frac{1}{2} + c_{\mathfrak{k},j} \right)$ and $c_{\mathfrak{k},j}$ is the eigenvalue of $C_{\mathfrak{k}}$ on \mathfrak{n}_j ;

(iii) $\text{Ric}(\mathfrak{p}, \mathfrak{n}) = 0$;

(iv) $\text{Ric}(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$, and $\text{Ric}(\mathfrak{n}_i, \mathfrak{n}_j) = 0$, if $i \neq j$.

We finally prove Theorem 1.3 given in Section 1.

Proof of Theorem 1.3: Let g_M be an adapted metric on M and $g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k})$ the associated $\text{Ad } L$ -invariant Euclidean product on \mathfrak{m} . By Corollary 3.7, we have that, if g_M is Einstein, then $C_{\mathfrak{p}}$ and $C_{\mathfrak{l}}$ are scalar on \mathfrak{n}_j , for every $j = 1, \dots, n$. Say

$$C_{\mathfrak{p}}|_{\mathfrak{n}_j} = b^j \text{Id} \text{ and } C_{\mathfrak{l}}|_{\mathfrak{n}_j} = c_{\mathfrak{l},j} \text{Id}.$$

Suppose that g is Einstein with constant E . From Corollary 4.3, we obtain the Einstein equations

$$(4.1) \quad -\frac{\lambda}{2\mu} b^j + r_j = \mu E, \quad j = 1, \dots, n$$

$$(4.2) \quad \frac{1}{2} \left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a} + \frac{\lambda^2}{2\mu^2} (1 - \gamma_a) \right) = \lambda E, \quad a = 1, \dots, s.$$

If $n > 1$, from Equation (4.1) we obtain the following:

$$(4.3) \quad \frac{\lambda}{2\mu} (b^i - b^j) = r_i - r_j, \quad i, j = 1, \dots, n.$$

By using Lemma 3.4 we have

$$r_i - r_j = \frac{1}{2} \left(\frac{1}{2} + c_{\mathfrak{k},i} \right) - \frac{1}{2} \left(\frac{1}{2} + c_{\mathfrak{k},j} \right) = \frac{1}{2} (c_{\mathfrak{k},i} - c_{\mathfrak{k},j}),$$

whereas

$$b^i - b^j = (c_{\mathfrak{k},i} - c_{\mathfrak{k},j}) - (c_{\mathfrak{l},i} - c_{\mathfrak{l},j}).$$

Therefore, Equation (4.3) becomes

$$-\frac{\lambda}{\mu} \underbrace{(c_{\mathfrak{l},i} - c_{\mathfrak{l},j})}_{\delta_{ij}^{\mathfrak{l}}} = \left(1 - \frac{\lambda}{\mu} \right) \underbrace{(c_{\mathfrak{k},i} - c_{\mathfrak{k},j})}_{\delta_{ij}^{\mathfrak{k}}}.$$

By using the variable X , we rewrite the equation above as $-\frac{1}{X} \delta_{ij}^{\mathfrak{l}} = \left(1 - \frac{1}{X} \right) \delta_{ij}^{\mathfrak{k}}$, and this yields $\delta_{ij}^{\mathfrak{l}} = (1 - X) \delta_{ij}^{\mathfrak{k}}$.

Equation (4.2) may be rewritten as

$$(4.4) \quad \frac{1}{2} \left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a} \right) X + (1 - \gamma_a) \frac{1}{4X} = \mu E.$$

Hence, if $s > 1$, for $a, b = 1, \dots, s$, we get

$$\frac{1}{2} \left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a} \right) X + (1 - \gamma_a) \frac{1}{4X} = \frac{1}{2} \left(\frac{\gamma_b}{2} + c_{\mathfrak{l},b} \right) X + (1 - \gamma_b) \frac{1}{4X},$$

which yields

$$\underbrace{c_{\mathfrak{l},a} - c_{\mathfrak{l},b}}_{\delta_{ab}^{\mathfrak{l}}} = \frac{1}{2} \left(\frac{1}{X^2} - 1 \right) \underbrace{(\gamma_a - \gamma_b)}_{\delta_{ab}^{\mathfrak{k}}}.$$

By solving this equation we obtain

$$(2\delta_{ab}^{\mathfrak{l}} + \delta_{ab}^{\mathfrak{k}})X^2 = \delta_{ab}^{\mathfrak{k}}.$$

Finally, by using Equations (4.1) and (4.4) we obtain the equality

$$\frac{1}{2} \left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a} \right) X + (1 - \gamma_a) \frac{1}{4X} = -\frac{b^j}{2X} + \frac{1}{2} \left(\frac{1}{2} + c_{\mathfrak{k},j} \right),$$

which rearranged gives

$$\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a} \right) X^2 - \left(\frac{1}{2} + c_{\mathfrak{k},j} \right) X + \frac{1}{2}(1 - \gamma_a + 2b^j) = 0.$$

□

Proof of Corollary 1.4: Since \mathfrak{p} is an irreducible $Ad L$ -module and \mathfrak{n} is an irreducible $Ad K$ -module, then any adapted metric on M is binormal. Hence, we use Theorem 1.3. By the irreducibility of \mathfrak{p} and \mathfrak{n} , we have $s = 1$ and $n = 1$ and thus Einstein binormal metrics are given by positive solutions of (1.9), if $C_{\mathfrak{p}}$ is scalar on \mathfrak{n} . Hence, from Theorem 1.3 we conclude that there exists on M an Einstein binormal metric if and only if $C_{\mathfrak{p}}$ is scalar on \mathfrak{n} and $\Delta \geq 0$, where

$$\Delta = (1 + 2c_{\mathfrak{k},\mathfrak{n}})^2 - 4(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})(1 - \gamma + 2b).$$

Since F is isotropy irreducible and $\dim F > 1$, we have $\gamma + 2c_{\mathfrak{l},p} \neq 0$ and the polynomial in (1.9) has exactly degree two. In fact, if $\gamma + 2c_{\mathfrak{l},p} = 0$, then $\gamma = c_{\mathfrak{l},p} = 0$ and thus, in particular, \mathfrak{p} lies in the center of \mathfrak{k} . But the hypothesis that \mathfrak{p} is irreducible and abelian implies that \mathfrak{p} is 1-dimensional which contradicts the hypothesis that $\dim F > 1$. Therefore, $\gamma + 2c_{\mathfrak{l},p} \neq 0$. In this case, the solutions of (1.9) are

$$X = \frac{1 + 2c_{\mathfrak{k},\mathfrak{n}} \pm \sqrt{\Delta}}{2(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})}.$$

□

Proof of Corollary 1.5: The fact that \mathfrak{p} is 1-dimensional implies that \mathfrak{p} lies in the center of \mathfrak{k} . Hence, in the notation of Corollary 1.4, $\gamma = c_{\mathfrak{l},\mathfrak{p}} = 0$. On the other hand, if \mathfrak{n} is $Ad K$ -irreducible then, the semisimple part of K acts transitively on \mathfrak{n} . Moreover, since \mathfrak{p} lies in the center of \mathfrak{k} , then the semisimple part of \mathfrak{l} coincides with the semisimple part of \mathfrak{k} . Hence, L also acts transitively on \mathfrak{n} and \mathfrak{n} is an irreducible $Ad L$ -module as well. Consequently, any G -invariant metric on M is adapted and moreover is binormal, by the irreducibility of \mathfrak{p} and \mathfrak{n} . Furthermore, $C_{\mathfrak{p}}$ must be scalar on \mathfrak{n} , since $C_{\mathfrak{k}}$ and $C_{\mathfrak{l}}$ are scalar on \mathfrak{n} . Therefore, G -invariant Einstein metrics are given by positive solutions of (1.9) in Theorem 1.3. Since $\gamma = c_{\mathfrak{l},\mathfrak{p}} = 0$, (1.9) is just a degree-one equation whose solution is

$$(4.5) \quad X = \frac{1 + 2b}{1 + 2c_{\mathfrak{k}, \mathfrak{n}}},$$

where b is the eigenvalue of $C_{\mathfrak{p}}$ on \mathfrak{n} . Since \mathfrak{g} is simple we have $\text{tr}(C_{\mathfrak{p}}) = \dim \mathfrak{p} = 1$. Since \mathfrak{p} lies in the center of \mathfrak{k} , $C_{\mathfrak{p}}$ vanishes on \mathfrak{k} and thus $\text{tr}(C_{\mathfrak{p}}) = \text{tr}(C_{\mathfrak{p}}|_{\mathfrak{n}}) = b \dim \mathfrak{n} = bm$. Hence, $b = 1/m$. By replacing b on (4.5) we obtain the expression given for X .

□

Proof of Corollary 1.6: If $\Phi \circ C_{\mathfrak{l}}|_{\mathfrak{p} \times \mathfrak{p}} = \alpha \Phi_{\mathfrak{k}}|_{\mathfrak{p} \times \mathfrak{p}}$, then

$$(4.6) \quad c_{\mathfrak{l}, a} = \alpha \gamma_a, \text{ for every } a = 1, \dots, s.$$

Therefore, for any $a, b = 1, \dots, s$, if $s > 1$, $2\delta_{ab}^{\mathfrak{l}} + \delta_{ab}^{\mathfrak{k}} = (2\alpha + 1)\delta_{ab}^{\mathfrak{k}}$ and, thus, Equation (1.8) in Theorem 1.3 becomes

$$(4.7) \quad (2\alpha + 1)\delta_{ab}^{\mathfrak{k}} X^2 = \delta_{ab}^{\mathfrak{k}}.$$

In particular, (4.6) implies that $c_{\mathfrak{l}, a} = 0$ if and only if $\gamma_a = 0$ (thus if \mathfrak{p} has submodules where L acts trivially, then $\Phi_{\mathfrak{k}}$ vanish on those submodules and then they lie in the center of \mathfrak{k} . If K is semisimple, then the isotropy representation of K/L is faithful). The fact that the isotropy representation of K/L is not irreducible implies that \mathfrak{p} decomposes as a direct sum $\mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_s$ with $s > 1$. For the indices for which $\gamma_a \neq \gamma_b$, we have $\delta_{ab}^{\mathfrak{k}} \neq 0$ and (4.7) implies that

$$X = \frac{1}{\sqrt{2\alpha + 1}}.$$

Hence, $X = \frac{1}{\sqrt{2\alpha + 1}}$ must be a root of the polynomial in (1.9). By using the fact that $c_{\mathfrak{l}, a} = \alpha \gamma_a$ and $b_j = c_{\mathfrak{k}, j} - c_{\mathfrak{l}, j}$, simple calculations show that

$$(4.8) \quad c_{\mathfrak{l}, j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right) \left(c_{\mathfrak{k}, j} + \frac{1}{2}\right).$$

We observe that this condition implies (1.7) in Theorem 1.3, as we can see by the equalities below:

$$\delta_{ij}^{\mathfrak{l}} = c_{\mathfrak{l}, i} - c_{\mathfrak{l}, j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right) (c_{\mathfrak{k}, i} - c_{\mathfrak{k}, j}) = (1 - X)\delta_{ij}^{\mathfrak{k}}.$$

Hence, there is a binormal Einstein metric if and only if (4.8) is satisfied and the operator $C_{\mathfrak{p}}$ is scalar on \mathfrak{n}_j , for every $j = 1, \dots, n$. In this case, according also to Theorem 1.3 such metric is, up to homothety, given by $B_{\mathfrak{p}} \oplus \frac{1}{\sqrt{2\alpha + 1}} B_{\mathfrak{n}}$.

It remains to show that if exists an Einstein binormal metric then $\sqrt{2\alpha + 1} \in \mathbb{Q}$. This follows from the fact that the eigenvalues of $C_{\mathfrak{k}}$ and $C_{\mathfrak{l}}$ on \mathfrak{n}_j are rational numbers. Since \mathfrak{k} is a compact algebra, the eigenvalue of its Casimir operator on the complex representation on $\mathfrak{n}_j^{\mathbb{C}}$ is given by

$$\frac{\langle \lambda_j, \lambda_j + 2\delta \rangle}{2h^*(\mathfrak{g})} \in \mathbb{Q},$$

where λ_j is the highest weight for $\mathfrak{n}_j^{\mathbb{C}}$, 2δ is the sum of all positive roots of \mathfrak{k} and $h^*(\mathfrak{g})$ is the dual Coxeter number of \mathfrak{g} ([11], [20]). A similar formula holds for $C_{\mathfrak{l}, j}$

and we conclude that $C_{\mathfrak{l},j}$ and $C_{\mathfrak{k},j}$ are rational numbers. If there exists a binormal Einstein metric on M , then $C_{\mathfrak{l},j}$ and $C_{\mathfrak{k},j}$ are related by formula (1.11) stated in this result. This implies that $\sqrt{2\alpha+1}$ is a rational number.

□

Example 4.1. Circle Bundles over Compact Irreducible Hermitian Symmetric Spaces:

An application of Corollary 1.5 occurs when the base space is an irreducible symmetric space. So let us consider a fibration $M \rightarrow N$ where the fiber F is isomorphic to the circle group and N is an isotropy irreducible symmetric space. Since F is the circle group, \mathfrak{p} lies in the center of \mathfrak{k} . Hence, K has one-dimensional center, since for a compact irreducible symmetric space the center of K has at most dimension 1. Moreover, in this case N is a compact irreducible Hermitian symmetric space. In particular, L must coincide with the semisimple part of K . Compact irreducible Hermitian symmetric spaces G/K are classified (see e.g. [10]). All the possible G , K and L are listed in Table 1, together with the coefficient X of the, unique, Einstein adapted metric on G/L , as in Corollary 1.5.

□

TABLE 1. Circle bundles over compact irreducible hermitian symmetric spaces.

G	K	L	X
$SU(n)$	$S(U(p) \times U(n-p))$	$SU(p) \times SU(n-p)$	$\frac{p(n-p)+1}{2p(n-p)}$
$SO(2n)$	$U(n)$	$SU(n)$	$\frac{n(n-1)+2}{2n(n-1)}$
$SO(n)$	$SO(2) \times SO(n-2)$	$SO(n-2)$	$\frac{n-1}{n-2}$
$Sp(n)$	$U(n)$	$SU(n)$	$\frac{n(n+1)+2}{2n(n+1)}$
E_6	$SO(10) \times U(1)$	$SO(10)$	$\frac{17}{32}$
E_7	$E_6 \times U(1)$	E_6	$\frac{14}{27}$

5. RIEMANNIAN FIBRATIONS WITH EINSTEIN FIBER AND EINSTEIN BASE

In this section we investigate conditions for the existence of an Einstein adapted metric g_M on M such that g_F and g_N are also Einstein. We prove Theorems 1.7 and 1.8 stated in Section 1. We follow the notation and hypothesis introduced in Section 1. In particular, for any homogeneous fibration $M = G/L \rightarrow G/K = N$, G is a compact connected semisimple Lie group and $L \subsetneq K \subsetneq G$ are connected closed non-trivial subgroups of G .

Proposition 5.1. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K = N$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

If g_M and g_N are both Einstein, then

$$\frac{\mu_j}{\mu_k} = \frac{r_j}{r_k} = \left(\frac{b^j}{b^k} \right)^{\frac{1}{2}}, \quad j, k = 1, \dots, n,$$

where b^j is the eigenvalue of the operator $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}$ on \mathfrak{n}_j , and the r_j 's are determined by $\text{Ric}^N = \oplus_{k=1}^n r_k B_{\mathfrak{n}_k}$ as in Lemma 3.4. Up to homothety, there exists at most one Einstein metric g_N on N such that the corresponding adapted metric g_M on M is Einstein.

Proof: The statement is trivial if N is isotropy irreducible, so we suppose that N is not irreducible, i.e., $n > 1$. From Corollary 3.7 we know that if g_M is Einstein, then there are constants b^j such that

$$\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a} |_{\mathfrak{n}_j} = b^j \text{Id}.$$

We recall from Lemma 3.4 that $\text{Ric}^N = \oplus_{k=1}^n r_k B_{\mathfrak{n}_k}$. Hence, if g_N is Einstein, then

$$(5.1) \quad \frac{r_1}{\mu_1} = \dots = \frac{r_n}{\mu_n}, \text{ i.e., } \frac{\mu_j}{\mu_k} = \frac{r_j}{r_k}, \text{ for } j, k = 1, \dots, n.$$

From Proposition 3.5, for $X \in \mathfrak{n}_k$, the Ricci curvature of g_M is

$$\text{Ric}(X, X) = -\frac{1}{2\mu_k} \sum_{a=1}^s \lambda_a B(C_{\mathfrak{p}_a} X, X) + r_k B(X, X) = \left(-\frac{b^k}{2\mu_k} + r_k \right) B(X, X).$$

If g_M is Einstein, then from the expression above we obtain the following Equations

$$(5.2) \quad -\frac{b^k}{2\mu_k^2} + \frac{r_k}{\mu_k} = -\frac{b^j}{2\mu_j^2} + \frac{r_j}{\mu_j}.$$

The identities (5.1) and (5.2) imply that

$$(5.3) \quad \frac{b^k}{\mu_k^2} = \frac{b^j}{\mu_j^2}$$

and consequently, by using (5.1),

$$\left(\frac{r_j}{r_k} \right)^2 = \left(\frac{\mu_j}{\mu_k} \right)^2 = \frac{b^j}{b^k}.$$

Since the eigenvalues b^j are independent of the μ_j 's, the ratios (5.3) imply that there is at most one possible choice for g_N , up to scalar multiplication.

□

Proposition 5.2. *Let g_M be an adapted metric on the homogeneous fibration $M = G/L \rightarrow G/K = N$ defined by the Euclidean product*

$$g_{\mathfrak{m}} = (\oplus_{a=1}^s \lambda_a B_{\mathfrak{p}_a}) \oplus (\oplus_{k=1}^n \mu_k B_{\mathfrak{n}_k}).$$

If g_M and g_F are both Einstein, then

$$\frac{\lambda_a}{\lambda_b} = \frac{q_a}{q_b} = \sum_{j=1}^n \frac{C_{n_j,b}}{\mu_j^2} \bigg/ \sum_{j=1}^n \frac{C_{n_j,a}}{\mu_j^2}, \text{ for } a, b = 1, \dots, s,$$

where $c_{n_j,a}$ is such that $\Phi(C_{n_j}, \cdot) \big|_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{n_j,a} \Phi \big|_{\mathfrak{p}_a \times \mathfrak{p}_a}$, for $a = 1, \dots, s$ and the q_a 's are determined by $\text{Ric}^F = \bigoplus_{a=1}^s q_a B_{\mathfrak{p}_a}$ as in Lemma 3.2. Up to homothety, there exists at most one Einstein metric g_F on F such that the corresponding adapted metric g_M on M is Einstein.

Proof: The statement is trivial if $F = K/L$ is isotropy irreducible, so we suppose that F is not irreducible, i.e., $s > 1$. The proof is similar to that of Proposition 5.1, by using Lemma 3.2 and Proposition 3.3.

□

Proof of Theorem 1.7: Using Proposition 5.1, we write $\mu_j^2 = \frac{b^j}{b^1} \mu_1^2$. The second formula follows immediately from this and Proposition 5.2.

□

Proof of Theorem 1.8: If g_M , g_N and g_F are all Einstein, from Proposition 3.5 we get

$$-\frac{1}{2\mu_j} b^j + \mu_j E_N = \mu_j E,$$

from which, if $E_N \neq E$, we deduce

$$(5.4) \quad \mu_j^2 = \frac{b^j}{2(E_N - E)}.$$

From Proposition 3.3, we have

$$\lambda_a E_F + \frac{\lambda_a^2}{4} \frac{C_{n_j,a}}{\mu_j^2} = \lambda_a E$$

which implies

$$(5.5) \quad \lambda_a \sum_{j=1}^n \frac{C_{n_j,a}}{\mu_j^2} = 4(E - E_F).$$

We obtain the required formula for λ_a by replacing (5.4) in the equation above.

□

6. INVARIANT EINSTEIN METRICS ON KOWALSKI n -SYMMETRIC SPACES

In this section we show that the n -symmetric spaces $M = \frac{G_0^n}{\Delta^n G_0}$ admit a non-standard Einstein adapted metric, for $n > 4$. The main result is Theorem 1.9 which is stated in section 1 and proved in this section.

Let G_0 be a compact connected simple Lie group. For any positive integer m , we denote $G_0^m = \underbrace{G_0 \times \dots \times G_0}_m$ and $\Delta^m G_0$ is the diagonal subgroup in G_0^m .

Let n, p, q be positive integers such that $p + q = n$ and $2 \leq p \leq q \leq n - 2$ and consider the following groups:

$$\begin{aligned}
(6.1) \quad & G = G_0^n \\
& K = \Delta^p G_0 \times \Delta^q G_0 \subsetneq G \\
& L = \Delta^n G_0 \subsetneq K
\end{aligned}$$

We consider the fibration $M = G/L \rightarrow G/K = N$ with fiber $F = K/L$. If \mathfrak{g}_0 denotes the Lie algebra of G_0 , then the Lie algebras of G , K and L are $\mathfrak{g} = \mathfrak{g}_0^n$, $\mathfrak{k} = \Delta^p \mathfrak{g}_0 \times \Delta^q \mathfrak{g}_0$ and $\mathfrak{l} = \Delta^n \mathfrak{g}_0$, respectively. If Φ_0 is the Killing form of \mathfrak{g}_0 , then the Killing form of \mathfrak{g} is simply $\Phi = \Phi_0 + \dots + \Phi_0$. Following the notation used in previous sections, let \mathfrak{n} be the orthogonal complement of \mathfrak{k} in \mathfrak{g} and \mathfrak{p} be an orthogonal complement of \mathfrak{l} in \mathfrak{k} , with respect to Φ . Then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n}$ and $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$.

In the Lemma below we present a decomposition of the orthogonal complements \mathfrak{p} and \mathfrak{n} . The tangent space to the fiber, \mathfrak{p} , is $Ad L$ -irreducible whereas \mathfrak{n} is $Ad K$ -reducible. The decomposition below for \mathfrak{n} is among many others and the decomposition we will consider throughout. The proof is straightforward and can be found in [3, §4.1].

Lemma 6.1. (i) $\mathfrak{p} = \{(\underbrace{qX, \dots, qX}_p, \underbrace{-pX, \dots, -pX}_q) : X \in \mathfrak{g}_0\}$ and \mathfrak{p} is $Ad L$ -irreducible;

(ii) \mathfrak{n} admits the decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where

$$\begin{aligned}
\mathfrak{n}_1 &= \{(X_1, \dots, X_p, 0, \dots, 0) : X_j \in \mathfrak{g}_0, \sum_{j=1}^p X_j = 0\} \subset \mathfrak{g}_0^p \times 0_q \\
\mathfrak{n}_2 &= \{(0, \dots, 0, X_1, \dots, X_q) : X_j \in \mathfrak{g}_0, \sum_{j=1}^q X_j = 0\} \subset 0_p \times \mathfrak{g}_0^q
\end{aligned}$$

Below we describe the Casimir operators of \mathfrak{g} , \mathfrak{k} , \mathfrak{l} , \mathfrak{p} , \mathfrak{n}_1 and \mathfrak{n}_2 and present the necessary eigenvalues to solve the Einstein equations for an Einstein adapted metric on M . The proofs of the following Lemmas can be found in [3, §4.1].

Lemma 6.2. (i) $C_{\mathfrak{g}} = Id_{\mathfrak{g}}$;

$$(ii) C_{\mathfrak{l}} = \frac{1}{n} Id_{\mathfrak{g}};$$

$$(iii) C_{\mathfrak{p}} = \frac{q}{np} Id_{\mathfrak{g}_0^p} \times \frac{p}{nq} Id_{\mathfrak{g}_0^q};$$

$$(iv) C_{\mathfrak{k}} = \frac{1}{p} Id_{\mathfrak{g}_0^p} \times \frac{1}{q} Id_{\mathfrak{g}_0^q};$$

$$(v) C_{\mathfrak{n}_1} = \left(1 - \frac{1}{p}\right) Id_{\mathfrak{g}_0^p} \times 0_{\mathfrak{g}_0^q} \text{ and } C_{\mathfrak{n}_2} = 0_{\mathfrak{g}_0^p} \times \left(1 - \frac{1}{q}\right) Id_{\mathfrak{g}_0^q}.$$

We recall that $c_{\mathfrak{l}, \mathfrak{p}}$ is the eigenvalue of $C_{\mathfrak{l}}$ on \mathfrak{p} , $c_{\mathfrak{k}, i}$ is the eigenvalue of $C_{\mathfrak{k}}$ on \mathfrak{n}_i . The Casimir operator of \mathfrak{p} is scalar on \mathfrak{n}_i , as we can see from Corollary 6.3, and b^i denotes the eigenvalue of $C_{\mathfrak{p}}$ on \mathfrak{n}_i , for $i = 1, 2$. Also, following a notation similar to that of (3.1) and (3.2), $c_{\mathfrak{n}_i, \mathfrak{p}}$ and γ are the constants defined by

$$(6.2) \quad \Phi(C_{\mathfrak{n}_i}, \cdot) |_{\mathfrak{p} \times \mathfrak{p}} = c_{\mathfrak{n}_i, \mathfrak{p}} \Phi |_{\mathfrak{p} \times \mathfrak{p}}, \quad i = 1, 2$$

$$(6.3) \quad \Phi(C_{\mathfrak{k}}, \cdot) |_{\mathfrak{p} \times \mathfrak{p}} = \gamma \Phi |_{\mathfrak{p} \times \mathfrak{p}}.$$

Lemma 6.3. (i) $c_{\mathfrak{l}, \mathfrak{p}} = \frac{1}{n}$;

(ii) $C_{\mathfrak{p}}$ is scalar on \mathfrak{n}_j , $j = 1, 2$ and $b^1 = \frac{q}{np}$ and $b^2 = \frac{p}{nq}$;

(iii) $c_{\mathfrak{k}, 1} = \frac{1}{p}$ and $c_{\mathfrak{k}, 2} = \frac{1}{q}$;

(iv) $\gamma = \frac{q^2 + p^2}{npq}$;

(v) $c_{\mathfrak{n}_1, \mathfrak{p}} = \frac{(p-1)q}{pn}$ and $c_{\mathfrak{n}_2, \mathfrak{p}} = \frac{(q-1)p}{qn}$.

We consider an adapted metric g_M on M defined by the Euclidean product

$$(6.4) \quad g_{\mathfrak{m}} = \lambda B_{\mathfrak{p}} \oplus \mu_1 B_{\mathfrak{n}_1} \oplus \mu_2 B_{\mathfrak{n}_2}.$$

We observe that \mathfrak{n}_1 and \mathfrak{n}_2 are inequivalent $Ad K$ -modules, but they are not irreducible, for $n > 4$. Hence, adapted metrics on M are not necessarily of the form (6.4). However, throughout we shall focus only on adapted metrics of this form.

We shall classify all the binormal Einstein metrics on M . Although the submodules \mathfrak{n}_1 and \mathfrak{n}_2 are not $Ad K$ -irreducible for $n > 4$, the Casimir operators of \mathfrak{l} , \mathfrak{k} and \mathfrak{p} are always scalar on \mathfrak{n}_1 and on \mathfrak{n}_2 . Hence, it is enough to consider one irreducible submodule in \mathfrak{n}_1 and one irreducible submodule in \mathfrak{n}_2 , to compute the Ricci curvature of g_M . According to Theorem 1.3 (2.19), there is an one-to-one correspondence, up to homothety, between binormal adapted Einstein metrics on M and positive solutions of the following set of equations:

$$(6.5) \quad \delta_{12}^{\mathfrak{k}}(1 - X) = \delta_{12}^{\mathfrak{l}}$$

$$(6.6) \quad (\gamma + 2c_{\mathfrak{l}, \mathfrak{p}})X^2 - (1 + 2c_{\mathfrak{k}, j})X + (1 - \gamma + 2b^j) = 0, \quad j = 1, 2$$

Given a positive solution X , then Einstein binormal metrics are given by $g_{\mathfrak{m}} = B_{\mathfrak{p}} \oplus X B_{\mathfrak{n}}$, up to homothety.

Theorem 6.4. *Let G_0 be a compact connected simple group and consider the fibration*

$$M = \frac{G_0^n}{\Delta^n G_0} \rightarrow \frac{G_0^p}{\Delta^p G_0} \times \frac{G_0^q}{\Delta^q G_0},$$

where $p + q = n$ and $2 \leq p \leq q \leq n - 2$.

If $p \neq q$ or $n = 4$, the standard metric is the only Einstein binormal metric, up to homothety. For $n > 4$ and $p = q$, there are on M exactly two Einstein binormal metrics, up to homothety, which are the standard metric and the metric induced by

$$g_{\mathfrak{m}} = B_{\mathfrak{p}} \oplus \frac{n}{4} B_{\mathfrak{n}}.$$

Proof: From Corollary 6.3 we obtain that $\delta_{12}^{\mathfrak{k}} = c_{\mathfrak{k}, 1} - c_{\mathfrak{k}, 2} = \frac{1}{p} - \frac{1}{q}$ whereas $\delta_{12}^{\mathfrak{l}} = c_{\mathfrak{l}, 1} - c_{\mathfrak{l}, 2} = \frac{1}{n} - \frac{1}{n} = 0$. Hence, Equation (6.5) implies that $X = 1$ or $p = q$. So if $p \neq q$, if there exists a binormal Einstein metric it must be the standard

metric. This we already know it is Einstein by [21]. Therefore, if $p \neq q$, then there exists, up to homothety, exactly one binormal Einstein metric on G/L which is the standard metric.

By using Corollary 6.3 Equation (6.6) may be rewritten as

$$(6.7) \quad nX^2 - q(p+2)X + pq + q - p = 0, \text{ for } j = 1$$

$$(6.8) \quad nX^2 - p(q+2)X + pq + p - q = 0, \text{ for } j = 2$$

It is clear that $X = 1$ is actually a solution of both equations, and we confirm that the standard metric is Einstein. Now suppose that $p = q$. As $n = p + q$, then $p = q = \frac{n}{2}$. Therefore, (6.7) and (6.8) become equivalent to $4X^2 - (n+4)X + n = 0$. This polynomial has two positive roots, 1 and $\frac{n}{4}$. Therefore, for $p = q$ and $n > 4$, there exist precisely two binormal Einstein metrics.

□

Since the vertical isotropy space \mathfrak{p} is $Ad L$ -irreducible, the restriction of any G -invariant metric on M to the fiber F is an Einstein metric. Below we show that the only Einstein adapted metric on M which projects onto an Einstein metric on the base space N is the Einstein binormal metric given in Theorem 6.4.

Theorem 6.5. *Let G_0 be a compact connected simple group and consider the fibration*

$$M = \frac{G_0^n}{\Delta^n G_0} \rightarrow \frac{G_0^p}{\Delta^p G_0} \times \frac{G_0^q}{\Delta^q G_0},$$

where $p + q = n$ and $2 \leq p \leq q \leq n - 2$. Let g_M be an Einstein adapted metric on M defined by $g_{\mathfrak{m}} = \lambda B_{\mathfrak{p}} \oplus \mu_1 B_{\mathfrak{n}_1} \oplus \mu_2 B_{\mathfrak{n}_2}$ as in (6.4). The projection g_N onto the base space is also Einstein if and only if $p = q$. In this case, g_M is a binormal metric.

Proof: By Proposition 5.1 we know that if g_M and g_N are Einstein then we must have the relation

$$(6.9) \quad \frac{r_1}{r_2} = \left(\frac{b^1}{b^2} \right)^{\frac{1}{2}}.$$

From Lemma 6.3 (ii) we obtain $\left(\frac{b^1}{b^2} \right)^{\frac{1}{2}} = \frac{a}{p}$. Since $[\mathfrak{n}_1, \mathfrak{n}_2] = 0$ and $C_{\mathfrak{l}}$ is scalar on \mathfrak{n}_i , it follows from the definition of the r_i 's in Proposition 3.5 and from Corollary 2.8 that $r_i = \frac{1}{2} \left(\frac{1}{2} + c_{\mathfrak{l},i} \right)$. Hence, $\frac{r_1}{r_2} = \frac{(p+2)q}{(q+2)p}$, by using Lemma 6.3 (iii). Therefore, (6.9) is possible if and only if $p = q$. Also from the proof of Proposition 5.1, if g_N and g_M are Einstein, then $\frac{\mu_1}{\mu_2} = \left(\frac{b^1}{b^2} \right)^{\frac{1}{2}} = \frac{p}{q} = 1$ and g_M is binormal. Conversely, if g_M is binormal and $p = q = n/2$, then by the above we also get

$$\frac{r_1}{\mu_1} = \frac{r_2}{\mu_2}$$

and g_N is Einstein.

□

The Einstein equations in general for arbitrary p and q are extremely complicated. However with the help of Maple it is still possible to solve the problem in general. Next we shall classify all the Einstein adapted metrics on M of the form

(6.4). The proof of the following Lemma follows from Proposition 4.1 and Corollary 2.8 by using the eigenvalues given in Corollary 6.3. This can be found in detail in [3, §4.1].

Lemma 6.6. *Consider the fibration*

$$M = \frac{G_0^n}{\Delta^n G_0} \rightarrow \frac{G_0^p}{\Delta^p G_0} \times \frac{G_0^q}{\Delta^q G_0} = N,$$

where $p + q = n$ and $2 \leq p \leq q \leq n - 2$. There is a one-to-one correspondence between Einstein adapted metrics on M defined by $g_m = \lambda B_p \oplus \mu_1 B_{n_1} \oplus \mu_2 B_{n_2}$ as in (6.4), up to homothety, and positive solutions of the following system of Equations:

$$(6.10) \quad -2q^2 X_1^2 + nq(p+2)X_1 + 2p^2 X_2^2 - np(q+2)X_2 = 0$$

$$(6.11) \quad n^2 + q^2(p+1)X_1^2 + p^2(q-1)X_2^2 - nq(p+2)X_1 = 0$$

To a positive solution (X_1, X_2) corresponds an Einstein adapted metric defined by $g_m = B_p \oplus \frac{1}{X_1} B_{n_1} \oplus \frac{1}{X_2} B_{n_2}$.

Proof of Theorem 1.9: By using Maple we obtain that the solutions of the system given in Lemma 6.6 are $X_1 = X_2 = 1$ and

$$(6.12) \quad X_1 = \alpha, X_2 = \left(\frac{-q^2(p+1)\alpha^2 + nq(p+2)\alpha - n^2}{p^2(q-1)} \right)^{\frac{1}{2}},$$

where α is a root of the polynomial

$$t(Z) = 4q^2 Z^3 - 4q(n + pq + 2)Z^2 + n(q(q+2)(p+1) + n + 8)Z - (q+3)n^2.$$

The solution $X_1 = X_2 = 1$ corresponds to a standard metric and, once more, we confirm that M is an Einstein standard manifold. We investigate the existence of other metrics. From the expression for X_2 in (6.12) we conclude that,

$$X_2 \in \mathbb{R} \text{ if and only if } \alpha \in \left(\frac{n}{q(p+1)}, \frac{n}{q} \right).$$

For this we compute the roots of the polynomial $-q^2(p+1)\alpha^2 + nq(p+2)\alpha - n^2$ in (6.12). Simple calculations show that

$$t\left(\frac{n}{q}\right) = \frac{p(q-1)^2 n^2}{q} > 0$$

$$t\left(\frac{n}{q(p+1)}\right) = -\frac{p(p+3)^2(q-1)n^2}{q(p+1)^3} < 0$$

and thus t has at least one (positive) root in the interval $\left(\frac{n}{q(p+1)}, \frac{n}{q}\right)$. To this root corresponds an Einstein adapted metric on M . Furthermore, we show that this root is unique and distinct from 1. From this we will conclude that there exists a non-standard Einstein adapted metric on M . Simple calculations show that the zeros of $\frac{dt}{dZ}$ are

$$\frac{n + pq + 2}{3q} \pm \frac{\sqrt{\delta}}{6q},$$

where

$$\delta = (q+1)^2 p^2 - (q-1)(3q^2 + 4q - 8)p - (q-1)(3q^2 + 8q + 16).$$

We show that $\delta < 0$. For $p = q$, $\delta = -2q^4 - 2q^3 + 8q^2 - 16q + 16 < 0$, for every $q \geq 2$. So we suppose that $p < q$. In this case, since $p^2 \leq (q-1)p$, we have

$$\begin{aligned} \delta &\leq (q-1)(-(2p+3)q^2 - (2p+8)q + (9p-16)) \\ &< (q-1)(-2p^3 - 5p^2 + p - 16) \\ &< 0, \end{aligned}$$

for every $p \geq 2$. With this we conclude that $\frac{dt}{dZ}$ has no real zeros and thus the root of t found above is the unique real root of t . Moreover, we must guarantee that this root does not yield the solution $X_1 = X_2 = 1$. If $X_1 = X_2 = 1$, then $\alpha = 1$ is a root of t . This may be possible since $1 \in \left(\frac{n}{q(p+1)}, \frac{n}{q}\right)$. Since

$$t(1) = p(q+2)(q-1)(n-4),$$

$\alpha = 1$ is a root of t if and only if $n = 4$. By using (6.12) we get that if $X_1 = 1$ when $n = 4$, then $X_2 = 1$ as well. Since non-standard Einstein adapted metrics are given by pairs of the form (6.12), with $\alpha \neq 1$, we conclude that there exists a unique non-standard Einstein adapted metric of the form (6.4) if and only if $n > 4$; in the case $n = 4$, the standard metric is the unique Einstein adapted metric of the form (6.4). Finally, we observe that, if $n = 4$, the subspaces \mathfrak{n}^1 and \mathfrak{n}^2 are irreducible $Ad L$ -submodules. Hence, any adapted metric on M is of the form (6.4). Therefore, we conclude that, for $n = 4$, there exists a unique Einstein adapted metric on M which is the standard metric.

Since there is a unique non-standard Einstein adapted metric on M , it follows from Theorem 6.4 that this metric is binormal if and only if $p = q$.

□

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